

Density theorems for bipartite graphs and related Ramsey-type results

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Abstract

In this paper, we present several density-type theorems which show how to find a copy of a sparse bipartite graph in a graph of positive density. Our results imply several new bounds for classical problems in graph Ramsey theory and improve and generalize earlier results of various researchers. The proofs combine probabilistic arguments with some combinatorial ideas. In addition, these techniques can be used to study properties of graphs with a forbidden induced subgraph, edge intersection patterns in topological graphs, and to obtain several other Ramsey-type statements.

1 Background and Introduction

Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is captured succinctly by the statement that “In a large system, complete disorder is impossible.” This is an area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to graph theory and combinatorics but also to logic, analysis, number theory, and geometry. Since the publication of the seminal paper of Ramsey in 1930, this subject has grown with increasing vitality, and is currently among the most active areas in combinatorics.

For a graph H , the *Ramsey number* $r(H)$ is the least positive integer n such that every two-coloring of the edges of complete graph K_n on n vertices, contains a monochromatic copy of H . Ramsey’s theorem states that $r(H)$ exists for every graph H . A classical result of Erdős and Szekeres [20], which is a quantitative version of Ramsey’s theorem, implies that $r(K_k) \leq 2^{2^k}$ for every positive integer k . Erdős [16] showed using probabilistic arguments that $r(K_k) > 2^{k/2}$ for $k > 2$. Over the last sixty years, there has been several improvements on these bounds (see, e.g., [13]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Determining or estimating Ramsey numbers is one of the central problem in combinatorics, see the book *Ramsey theory* [27] for details. Besides the complete graph, the next most classical topic in this area concerns the Ramsey numbers of sparse graphs, i.e., graphs with certain upper bound constraints on the degrees of the vertices. The study of these Ramsey numbers was initiated by Burr and Erdős in 1975, and this topic has since placed a central role in graph Ramsey theory.

An *induced subgraph* is a subset of the vertices of a graph together with all edges whose both endpoints are in this subset. There are several results and conjectures which indicate that graphs which do not contain a fixed induced subgraph are highly structured. In particular, the most famous conjecture of this sort by Erdős and Hajnal [18] says that every graph G on n vertices which does not contain a fixed induced subgraph H has a clique or independent set of size a power of n . This is in

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striking contrast with the general case where one can not guarantee a clique or independent set of size larger than logarithmic in the number of vertices.

Results in Ramsey theory generally say that if a large enough structure is partitioned into a small number of parts, then one of the resulting parts will contain some desired substructure. Sometimes, a stronger *density-type* result can be proved, which shows that any dense subset of a large enough structure contains the desired substructure. One famous example is Szemerédi's theorem, which says that every subset of the positive integers of positive upper density contains arbitrarily long arithmetic progressions. It strengthens the earlier result of van der Waerden that every finite partition of the positive integers contain arbitrarily long arithmetic progressions, and has led to many deep and beautiful results in various areas of mathematics, including the recent spectacular result of Green and Tao that there are arbitrarily long arithmetic progressions in primes.

It is easy to see that Ramsey's theorem has no density-type analogue. Indeed, the complete bipartite graph with both parts of size $n/2$ has $n^2/4$ edges, i.e., more than half the total possible number of edges, and still does not contain a triangle. However, for bipartite graphs, a density version exists as was shown by Kövari, Sós, and Turán [36] in 1954.

In this paper, we present several density-type theorems which show how to find a copy of a sparse bipartite graph in a graph of positive density. Our results imply several new bounds for classical problems in graph Ramsey theory and improve and generalize earlier results of various researchers. The proofs combine probabilistic arguments with some combinatorial ideas. In addition, these techniques can be used to study edge intersection patterns in topological graphs, make some progress towards the Erdős-Hajnal conjecture, and obtain several other Ramsey-type statements. In the subsequent sections we present in full detail our theorems and compare them with previously obtained results.

1.1 Ramsey numbers and density-type theorems for bipartite graphs

Estimating Ramsey numbers is one of the central (and difficult) problems in modern combinatorics. Among the most interesting questions in this area are the linear bounds for Ramsey numbers of graphs with certain degree constraints. In 1975, Burr and Erdős [8] conjectured that, for each positive integer Δ , there is a constant $c(\Delta)$ such that every graph H with n vertices and maximum degree Δ satisfies $r(H) \leq c(\Delta)n$. This conjecture was proved by Chvátal, Rödl, Szemerédi, and Trotter [11]. Their proof is a beautiful illustration of the power of Szemerédi's regularity lemma [30]. However, the use of this lemma makes an upper bound on $c(\Delta)$ to grow as a tower of 2s with height polynomial in Δ . Since then, the problem of determining the correct order of magnitude of $c(\Delta)$ as a function of Δ has received considerable attention from various researchers. Still using a variant of the regularity lemma, Eaton [15] showed that $c(\Delta) < 2^{2^{c\Delta}}$ for some fixed c . A novel approach of Graham, Rödl, and Rucinski [25] gave the first linear upper bound on Ramsey numbers of bounded degree graphs without using any form of the regularity lemma. Their proof implies that $c(\Delta) < 2^{c\Delta \log^2 \Delta}$. (Here, and throughout the paper, all logarithms are base 2.)

The case of bipartite graphs with bounded degree was studied by Graham, Rödl, and Rucinski more thoroughly in [26], where they improved their upper bound, showing that $r(H) \leq 2^{\Delta \log \Delta + O(\Delta)} n$ for every bipartite graph H with n vertices and maximum degree Δ . As they point out, their proof does not give a stronger density-type result. In the other direction, they proved that there is a positive constant c such that, for every $\Delta \geq 2$ and $n \geq \Delta + 1$, there is a bipartite graph H with n vertices and maximum degree Δ satisfying $r(H) \geq 2^{c\Delta} n$. Closing the gaps between these two bounds remained a challenging open problem. In this paper, we solve this problem by showing that the correct order of

magnitude of the Ramsey number of bounded degree bipartite graphs is essentially given by the lower bound. This follows from the following density-type theorem.

Theorem 1.1 *Let H be a bipartite graph with n vertices and maximum degree $\Delta \geq 1$. If $\epsilon > 0$ and G is a graph with $N \geq 32\Delta\epsilon^{-\Delta}n$ vertices and at least $\epsilon\binom{N}{2}$ edges, then H is a subgraph of G .*

Taking $\epsilon = 1/2$ together with the majority color in a 2-coloring of the edges of K_N , we obtain a corollary which gives a best possible upper bound up to the constant factor in the exponent on Ramsey numbers of bounded degree bipartite graphs.

Corollary 1.2 *If H is bipartite, has n vertices and maximum degree $\Delta \geq 1$, then $r(H) \leq \Delta 2^{\Delta+5}n$.*

Moreover, the above theorem also easily gives an upper bound on multicolor Ramsey numbers of bipartite graphs. The k -color Ramsey number $r(H_1, \dots, H_k)$ is the least positive integer N such that for every k -coloring of the edges of the complete graph K_N , there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k$. Taking $\epsilon = 1/k$ in Theorem 1.1 and considering the majority color in a k -coloring of the edges of a complete graph shows that for bipartite graphs H_1, \dots, H_k each with n vertices and maximum degree at most Δ , $r(H_1, \dots, H_k) \leq 32\Delta k^{\Delta}n$.

One family of bipartite graphs that have received particular attention are the d -cubes. The d -cube Q_d is the d -regular graph with 2^d vertices whose vertex set is $\{0, 1\}^d$ and two vertices are adjacent if they differ in exactly one coordinate. Burr and Erdős conjectured that $r(Q_d)$ is linear in the number of vertices of the d -cube. Beck [6] proved that $r(Q_d) \leq 2^{cd^2}$. The bound of Graham et al. [25] gives the improvement $r(Q_d) \leq 8(16d)^d$. Shi [43], using ideas of Kostochka and Rödl [32], proved that $r(Q_d) \leq 2^{(\frac{3+\sqrt{5}}{2}d+o(d))}$, which is a polynomial bound in the number of vertices with exponent $\frac{3+\sqrt{5}}{2} \approx 2.618$. A very special case of Corollary 1.2, when $H = Q_d$, gives immediately the following improved result.

Corollary 1.3 *For every positive integer d , $r(Q_d) \leq d2^{2d+5}$.*

A graph is d -degenerate if every subgraph of it has a vertex of degree at most d . Notice that graphs with maximum degree d are d -degenerate. This notion nicely captures the concept of sparse graphs as every t -vertex subgraph of a d -degenerate graph has at most td edges. (Indeed, remove from the subgraph a vertex of minimum degree, and repeat this process in the remaining subgraph.) Burr and Erdős [8] conjectured that, for each positive integer d , there is a constant $c(d)$ such that $r(H) \leq c(d)n$ for every d -degenerate graph H on n vertices. This well-known and difficult conjecture is a substantial generalization of the above mentioned results on Ramsey numbers of bounded degree graphs and progress on this problem was made only recently.

Kostochka and Rödl [33] were the first to prove a polynomial upper bound on the Ramsey numbers of d -degenerate graphs. They showed that $r(H) \leq c_d n^2$ for every d -degenerate graph H with n vertices. A nearly linear bound of the form $r(H) \leq c_d n^{1+\epsilon}$ for any fixed $\epsilon > 0$ was obtained in [35]. For bipartite H , Kostochka and Rödl proved that $r(H) \leq d^{d+o(d)}\Delta n$, where Δ is the maximum degree of H . Kostochka and Sudakov [35] proved that $r(H) \leq 2^{O(\log^{2/3} n)}n$ for every d -degenerate bipartite graph H with n vertices and constant d . Here we improve on both of these results.

Theorem 1.4 *If $d/n \leq \delta \leq 1$, H is a d -degenerate bipartite graph with n vertices and maximum degree $\Delta \geq 1$, G is a graph with N vertices and at least $\epsilon\binom{N}{2}$ edges, and $N \geq 2^{12}\epsilon^{-(1/\delta+3)d-2}\Delta^{\delta}n$, then H is a subgraph of G .*

For δ and H as in the above theorem, taking $\epsilon = 1/2$ and considering the majority color in a 2-coloring of the edges of K_N shows that

$$r(H) \leq 2^{\delta^{-1}d+3d+14} \Delta^\delta n.$$

This new upper bound on Ramsey numbers for bipartite graphs is quite versatile. Taking $\delta = 1$, we have $r(H) \leq 2^{4d+14} \Delta n$ for bipartite d -degenerate graphs with n vertices and maximum degree Δ . This improves upon the bound of Kostochka and Rödl. If $\Delta \geq 2^d$, then taking $\delta = (\frac{d}{\log \Delta})^{1/2}$, we have

$$r(H) \leq 2^{2\sqrt{d \log \Delta} + 3d + 14} n$$

for bipartite d -degenerate graphs H with n vertices and maximum degree Δ . In particular, we have $r(H) \leq 2^{O(\log^{1/2} n)} n$ for constant d . This improves on the bound of Kostochka and Sudakov, and is another step closer to the Burr-Erdős conjecture.

Moreover, as long as Δ is at most exponential in d , we still have $r(H) \leq 2^{O(d)} n$. This has interesting applications to another notion of sparseness introduced by Chen and Schelp [10]. A graph is p -arrangeable if there is an ordering v_1, \dots, v_n of the vertices such that for any vertex v_i , its neighbors to the right of v_i have together at most p neighbors to the left of v_i (including v_i). This is an intermediate notion of sparseness not as strict as bounded degree though not as general as bounded degeneracy. Extending the result of [11], Chen and Schelp proved that there is a constant $c(p)$ such that every p -arrangeable graph H on n vertices has Ramsey number at most $c(p)n$. This gives linear Ramsey numbers for planar graphs and more generally for graphs that can be drawn on a bounded genus surfaces. The best known bound [25] for p -arrangeable bipartite H is $r(H) \leq 2^{cp \log p} n$, where c is a constant. The proof of Theorem 1.4 can be modified to give $r(H) \leq 2^{cp} n$ for every p -arrangeable bipartite graph H , which is an essentially best possible bound. Note that for every vertex v_i in a p -arrangeable graph, there is a subset $S_i \subset \{v_1, \dots, v_{i-1}\}$ of size at most $p-1$ such that for any vertex $v_j, j > i$ adjacent to v_i , its neighbors in $\{v_1, \dots, v_{i-1}\}$ form a subset of S_i . Therefore, there are at most 2^{p-1} distinct such subsets of neighbors. This important observation essentially allows us to treat p -arrangeable bipartite graphs as if they were p -degenerate graphs with maximum degree at most 2^{p-1} , which in turn gives the above bound on Ramsey numbers.

In spite of the above mentioned progress, the Burr-Erdős conjecture is still open even for the special case of d -degenerate bipartite graphs in which every vertex in one part has degree at most $d \geq 3$. Using our approach, one can make some progress on this special case, which is discussed in the concluding remarks.

It seems plausible that $r(H) \leq 2^{c\Delta} n$ holds in general for every graph H with n vertices and maximum degree Δ . The following result shows that this is at least true for graphs of bounded chromatic number.

Theorem 1.5 *If H has n vertices, chromatic number q , and maximum degree Δ , then $r(H) \leq 2^{4q\Delta} n$.*

1.2 Subgraph Multiplicity

Recall that Ramsey's theorem states that every 2-edge-coloring of a sufficiently large complete graph K_N contains at least one monochromatic copy of a given graph H . Let $c_{H,N}$ denote the fraction of copies of H in K_N that must be monochromatic in any 2-edge-coloring. By an averaging argument, $c_{H,N}$ is a bounded, monotone increasing function in N , and therefore has a limit c_H as $N \rightarrow \infty$. The

constant c_H is known as the *Ramsey multiplicity constant* for the graph H . It is simple to show for H with m edges that $c_H \leq 2^{1-m}$, where this bound comes from considering a random 2-edge-coloring of K_N with each coloring equally likely.

Erdős and in a more general form Burr and Rosta suggested that the Ramsey multiplicity constant is achieved by a random coloring. These conjectures are false as was demonstrated Thomason [48] even for H being any complete graph K_n with $n \geq 4$. Moreover, as shown in [21], there are H with m edges and $c_H \leq m^{-m/2+o(m)}$, which demonstrates that the random coloring is far from being optimal for some graphs.

For bipartite graphs the situation seems to be very different. The *edge density* of a graph is the fraction of pairs of vertices that are edges. The conjectures of Simonovits [45] and Sidorenko [44] suggest that for any bipartite H the number of its copies in any graph G on N vertices and edge density ϵ ($\epsilon > N^{-\gamma(H)}$) is asymptotically at least the same as in the N -vertex random graph with edge density ϵ . So far it is known only in very special cases, i.e., for complete bipartite graphs, trees, even cycles (see [44]), and recently for cubes [28]. Our Theorem 1.1 can be strengthened as follows to give additional evidence for the validity of this conjecture.

Theorem 1.6 *Let H be a bipartite graph with n vertices and maximum degree $d \geq 1$. If $\epsilon > 0$ and G is a graph with $N \geq 32d\epsilon^{-d}n$ vertices and at least $\epsilon \binom{N}{2}$ edges, then G contains at least $(2^7 d)^{-n/2} \epsilon^{dn} N^n$ labeled copies of H .*

Notice that this theorem roughly says that a large graph with edge density ϵ contains at least ϵ^{dn} fraction of all possible copies of H . If H is d -regular, i.e., has $dn/2$ edges, then the random graph with edge density ϵ contains $\epsilon^{dn/2}$ fraction of all possible copies of H . This shows that for regular H the exponent of ϵ in the above theorem is only by a factor 2 away from the conjectured bound. Moreover, the same is true with a different factor for every d -degenerate bipartite graph H with maximum degree at most exponential in d . This follows from an extension of our result on d -degenerate bipartite graphs which is discussed in Section 3. A similar extension for graphs with bounded chromatic number is discussed in Section 4.

1.3 Subdivided subgraphs in dense graphs

A *topological copy* of a graph H is any graph formed by replacing edges of H by internally vertex disjoint paths. This is an important notion in graph theory, e.g., the celebrated theorem of Kuratowski uses it to characterize planar graphs. In the special case in which each of the paths replacing edges of H has length $t + 1$, we obtain a *t -subdivision* of H . An old conjecture of Mader and Erdős-Hajnal which was proved in [7, 31] says that there is a constant c such that every graph with n vertices and at least cp^2n edges contains a topological copy of K_p .

Erdős [17] asked whether every graph on n vertices with $c_1 n^2$ edges contains a 1-subdivision of a complete graph K_m with $m \geq c_2 \sqrt{n}$ for some constant c_2 depending on c_1 . Note that the above mentioned result implies that any such graph on n vertices will contain a topological copy of a complete graph on $\Omega(\sqrt{n})$ vertices, but not necessarily a 1-subdivision. The existence of such a subdivision was proved in [3], giving a positive answer to the question of Erdős. Note that clique of order $O(\sqrt{n})$ has $O(n)$ edges. So it is natural to ask whether the conjecture of Erdős can be generalized to show that under the same conditions as above one can find a 1-subdivision of every graph with $O(n)$ edges, not just of a clique.

A result closely related to this question was obtained by Alon et al. in [2] (see also [30]). They proved, using Szemerédi's regularity lemma, that any graph with n vertices and at least $c_1 n^2$ edges contains a topological copy of every graph with at most $c_2 n$ edges (c_2 depends on c_1). Moreover, their proof shows that the topological copy of H can be taken to be a 3-subdivision of H .

Motivated by the conjecture of Burr and Erdős that graphs with bounded degeneracy have linear Ramsey numbers, Alon [1] proved that any graph on n vertices in which no two vertices of degree at least three are adjacent has Ramsey number at most $12n$. In particular, the Ramsey number of a 1-subdivision of an arbitrary graph with n edges is linear in n .

The following density-type theorem improves on these previous results concerning subdivided graphs, and gives a positive answer to the generalization of the Erdős conjecture mentioned above.

Theorem 1.7 *Let H be a graph with n edges and no isolated vertices and let G be a graph with N vertices and ϵN^2 edges such that $N \geq 100\epsilon^{-3}n$. Then G contains the 1-subdivision of H .*

1.4 Forbidden induced subgraphs

A graph is H -free if it does not contain H as an *induced* subgraph. A basic property of large random graphs is that they almost surely contain any fixed graph H as an induced subgraph. Therefore, there is a general belief that H -free graphs are highly structured. For example, Erdős and Hajnal [18] proved that every H -free graph on N vertices contains a clique or independent set of size at least $2^{c\sqrt{\log N}}$, where $c > 0$ only depends on H . This is in striking contrast with the general case where one can not guarantee a clique or independent set of size larger than logarithmic in N . Erdős-Hajnal further conjectured that this bound can be improved to N^c . This famous conjecture has only been solved for some particular H (see, e.g. [4] and [12]).

An interesting partial result for the general case was obtained by Erdős, Hajnal, and Pach [19]. They show that every H -free graph G with N vertices or its complement \bar{G} contains a complete bipartite graph with parts of size $N^{c(H)}$. We obtain a strengthening of this result which brings it closer to the Erdős-Hajnal conjecture.

Theorem 1.8 *For every graph H , there is $c > 0$ such that any H -free graph on N vertices contains a complete bipartite graph with parts of size N^c or an independent set of size N^c .*

To get a better understanding of the properties of H -free graphs, one can naturally ask for an *asymmetric* version of the Erdős-Hajnal result. The proof in [18] first shows that every H -free graph G on N vertices contains a *perfect* induced subgraph of order $2^{c\sqrt{\log N}}$. It then uses a well known fact that every perfect graph on n vertices contains a clique or an independent set of order \sqrt{n} . Therefore, it is not clear how to adjust this proof to improve the bound of $2^{c\sqrt{\log N}}$ in the case when we know that the maximum clique or independent set in G is rather small. The general framework we develop in this paper can be used to obtain such a generalization of the Erdős-Hajnal result.

Theorem 1.9 *There exists $c = c(H) > 0$ such that for any H -free graph G on N vertices and n_1, n_2 satisfying $(\log n_1)(\log n_2) \leq c \log N$, G contains a clique of size n_1 or an independent set of size n_2 .*

1.5 Edge intersection patterns in topological graphs

The origins of graph theory are closely connected with topology and geometry. Indeed, the first monograph on graph theory, by König in 1935, was entitled *Combinatorial Topology of Systems of Segments*. In recent years, geometric graph theory, which studies intersection patterns of geometric objects and graph drawings, has rapidly developed.

A *topological graph* is a graph drawn in the plane with vertices as points and edges as curves connecting endpoints and passing through no other vertices. A topological graph is *simple* if any two edges have at most one point in common. A very special case of simple topological graphs is *geometric graphs* in which edges are straight-line segments. There are many well known open problems about graph drawings and in particular edge intersection patterns of topological graphs. Even some innocent looking questions in this area can be quite difficult.

For example, more than 40 years ago Conway asked what is the maximum size of a *thrackle*, that is, a simple topological graph in which every two edges intersect. He conjectured that every n -vertex thrackle has at most n edges. Lovász, Pach, and Szegedy [38] were the first to prove a linear upper bound on the number of edges in a thrackle, and despite some improvement in [9], the conjecture is still open. On the other hand, Pach and Tóth [41] constructed drawings of the complete graph in the plane with each pair of edges having at least one and at most two points in common. Hence, to ensure a pair of disjoint edges, the assumption that the topological graph is simple is necessary.

For dense simple topological graphs, one might expect to obtain a much stronger conclusion than that of Conway's conjecture, showing that these graphs contain large patterns of pairwise disjoint edges. Our next theorem proves that this is indeed true, extending an earlier result of Pach and Solymosi [40] for geometric graphs.

Theorem 1.10 *For each $\gamma > 0$ there is $\delta > 0$ and n_0 such that every simple topological graph $G = (V, E)$ with $n \geq n_0$ vertices and $m \geq \gamma n^2$ edges contains two disjoint edge subsets E_1, E_2 each of cardinality at least δn^2 such that every edge in E_1 is disjoint from every edge in E_2 .*

This result has a natural interpretation in the context of Ramsey theory for intersection graphs. The *intersection graph* of a collection of curves in the plane has a vertex for each curve and two of its vertices are adjacent if their corresponding curves intersect. It is easy to show that the 1-subdivision of K_5 is not an intersection graph of curves in the plane and thus the edge intersection graph of a topological graph has a fixed forbidden induced subgraph. Therefore, the properties of intersection graphs are closely related to the Erdős-Hajnal conjecture mentioned in the previous subsection, and one might expect to find in these graphs two large vertex subsets with no edges between them. Nevertheless, Theorem 1.10 is still quite surprising because it shows that the edge intersection graph of any dense simple topological graph contains two *linear*-sized subsets with no edges between them.

Another interesting Ramsey-type problem is to estimate the maximum number of pairwise disjoint edges in any complete simple topological graph. Pach and Tóth [41] proved that every simple topological graph of order n without k pairwise disjoint edges has $O(n(\log n)^{4k-8})$ edges. They use this to show that every complete simple topological graph of order n has $\Omega(\log n / \log \log n)$ pairwise disjoint edges. Using Theorem 1.10, we give a modest improvement on this bound (the truth here is probably n^ϵ). Our result is valid for dense (not only complete) simple topological graphs as well.

Corollary 1.11 *There is $\epsilon > 0$ such that every complete simple topological graph of order n contains $\Omega((\log n)^{1+\epsilon})$ pairwise disjoint edges.*

The proof of the above two results rely on a new theorem concerning the edge distribution of H -free graphs. It extends earlier results of [42] and [23] which show that H -free graphs contain large induced subgraphs that are very sparse or dense. However, these results are not sufficient for our purposes. We prove that H -free graphs satisfying a seemingly weak edge density condition contain a very dense linear-sized induced subgraph.

1.6 Induced Ramsey numbers

In the early 1970's an important generalization of Ramsey's theorem, the Induced Ramsey Theorem, was discovered independently by Deuber; Erdős, Hajnal, and Posa; and Rödl. We write

$$G \xrightarrow{\text{ind}} (H_1, \dots, H_k)$$

if, for every k -coloring of the edges of G with colors $1, \dots, k$, there is an index i and an induced copy of H_i in G that is monochromatic of color i . The Induced Ramsey Theorem states that for all graphs H_1, \dots, H_k , there is a graph G such that $G \xrightarrow{\text{ind}} (H_1, \dots, H_k)$, and the *induced Ramsey number* $r_{\text{ind}}(H_1, \dots, H_k)$ is the minimum number of vertices in such G . If all $H_i = H$, then we denote $r_{\text{ind}}(H_1, \dots, H_k) = r_{\text{ind}}(H; k)$.

Early proofs of the Induced Ramsey Theorem give weak bounds on these numbers. For two colors, the more recent results [23], [29] significantly improve these estimates. However, it seems that the approaches in those papers do not generalize to give good results for many colors. There is a simple way of giving an upper bound on the multicolor induced Ramsey $r_{\text{ind}}(H_1, \dots, H_k)$ in terms of induced Ramsey numbers with fewer colors. Notice that if $G_1 \xrightarrow{\text{ind}} (H_1, \dots, H_\ell)$, $G_2 \xrightarrow{\text{ind}} (H_{\ell+1}, \dots, H_k)$, and $G \xrightarrow{\text{ind}} (G_1, G_2)$, then $G \xrightarrow{\text{ind}} (H_1, \dots, H_k)$. (To see this, just group together the first ℓ colors and the last $k - \ell$ colors.) For fixed H , this gives that $r_{\text{ind}}(H; k)$ grows at most like a tower of 2s of height roughly $\log k$. The following result improves considerably on this tower bound.

Theorem 1.12 *For every graph H there is a constant $c(H)$ such that $r_{\text{ind}}(H; k) \leq k^{c(H)k}$ for every integer $k \geq 2$.*

For H on n vertices, the proof shows that $c(H)$ can be taken to be $500n^3$. It is worth mentioning that as a function of k (up to the constant $c(H)$), the upper bound in Theorem 1.12 is similar to the best known estimate for ordinary Ramsey numbers. On the other hand, it is known and easy to show that in general these numbers grow at least exponentially in k . The proof of the above theorem combines ideas used to establish bounds on Ramsey numbers of graphs with bounded chromatic number together with some properties of pseudo-random graphs.

Organization of the paper. In the next section we present our key ideas and techniques and illustrate them on a simple example, the proof of Theorem 1.6. More involved applications of these techniques which require additional ideas are given in Sections 3-5. There we prove results on bipartite degenerate graphs, graphs with bounded chromatic number, and subdivided graphs, respectively. In Section 6, we prove a useful embedding lemma for induced subgraphs which we apply in Section 7 together with our basic techniques to obtain two results on the Erdős-Hajnal conjecture. In Section 8, we apply this lemma again to show that H -free graphs satisfying a rather weak edge density condition contain a very dense linear-sized induced subgraph. We then use this fact about H -free graphs in Section 9 to prove two results on disjoint edge patterns in simple topological graphs. In Section 10,

we prove Theorem 1.12 which gives an upper bound on multicolor induced Ramsey numbers. The last section of this paper contains some concluding remarks together with a few conjectures and open problems. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

2 Dependent random choice and graph embeddings

The purpose of this section is to illustrate on the simplest example, the proof of Theorem 1.6, the key ideas and techniques that we will use. The first tool is a simple yet surprisingly powerful lemma whose proof uses a probabilistic argument known as *dependent random choice*. Early versions of this technique were developed in the papers [24, 32, 46]. Later, variants were discovered and applied to various Ramsey and density-type problems (see, e.g., [35, 3, 47, 34]).

This lemma demonstrates that every dense graph contains a large set of vertices A with the useful property that *almost all* small subsets of A have many common neighbors. The earlier applications of dependent random choice for Ramsey-type problems (e.g., [32, 46, 35, 3, 34]) required that *all* small subsets of A have large common neighborhood. This stronger assumption, which is possible to obtain using dependent random choice, allows one to use a simple greedy procedure to embed sparse graphs. However, the price of achieving this stronger property is rather high, since the resulting set A has a sublinear number of vertices in the order of the graph. Consequently, one cannot use this to prove a linear upper bound on Ramsey numbers. Our main contribution here shows how to circumvent this difficulty. The second tool, Lemma 2.2, is an embedding result for hypergraphs. It can be used to embed sparse bipartite graphs without requiring all subsets of A to have large common neighborhood.

For a vertex v in a graph G , let $N(v)$ denote the set of neighbors of v in G . Given a subset $U \subset G$, the *common neighborhood* $N(U)$ of U is the set of all vertices of G that are adjacent to U , i.e., to *every* vertex in U . Sometimes, we write $N_G(U)$ to stress that the underlying graph is G when this is not entirely clear from the context. By a d -set, we mean a set of cardinality d . The following lemma demonstrates that every dense bipartite graph contains a large set of vertices A such that almost every d -set in A has many common neighbors.

Lemma 2.1 *If $\epsilon > 0$ and $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = N$ and at least ϵN^2 edges, then for all positive integers a, d, t, x , there is a subset $A \subset V_2$ with $|A| \geq 2^{-1/a} \epsilon^t N$ such that for all but at most $2\epsilon^{-ta} \left(\frac{x}{N}\right)^t \left(\frac{|A|}{N}\right)^a \binom{N}{d}$ d -sets S in A , we have $|N(S)| \geq x$.*

Proof. Let T be a subset of t random vertices of V_1 , chosen uniformly with repetitions. Set $A = N(T)$, and let X denote the cardinality of $A \subset V_2$. By linearity of expectation and by convexity of $f(z) = z^t$,

$$\mathbb{E}[X] = \sum_{v \in V_2} \left(\frac{|N(v)|}{N} \right)^t = N^{-t} \sum_{v \in V_2} |N(v)|^t \geq N^{1-t} \left(\frac{\sum_{v \in V_1} |N(v)|}{N} \right)^t \geq \epsilon^t N.$$

Let Y denote the random variable counting the number of d -sets in A with fewer than x common neighbors. For a given d -set S , the probability that S is a subset of A is $\left(\frac{|N(S)|}{N}\right)^t$. Therefore, we have

$$\mathbb{E}[Y] \leq \binom{N}{d} \left(\frac{x-1}{N} \right)^t.$$

By convexity, $\mathbb{E}[X^a] \geq \mathbb{E}[X]^a$. Thus, using linearity of expectation, we obtain

$$\mathbb{E} \left[X^a - \frac{\mathbb{E}[X]^a}{2\mathbb{E}[Y]} Y - \frac{\mathbb{E}[X]^a}{2} \right] \geq 0.$$

Therefore, there is a choice of T for which this expression is nonnegative. Then

$$X^a \geq \frac{1}{2} \mathbb{E}[X]^a \geq \frac{1}{2} \epsilon^{ta} N^a$$

and

$$Y \leq 2X^a \mathbb{E}[Y] \mathbb{E}[X]^{-a} < 2\epsilon^{-ta} \left(\frac{x}{N} \right)^t \left(\frac{|A|}{N} \right)^a \binom{N}{d}.$$

This implies $|A| = X \geq 2^{-1/a} \epsilon^t N$, completing the proof. \square

A *hypergraph* $\mathcal{F} = (V, E)$ consists of a vertex set V and an edge set E , which is a collection of subsets of V . It is *down-closed* if $e_1 \subset e_2$ and $e_2 \in E$ implies $e_1 \in E$. The following lemma shows how to embed a sparse hypergraph in a very dense hypergraph.

Lemma 2.2 *Let \mathcal{H} be a n -vertex hypergraph with maximum degree d such that each edge of \mathcal{H} has size at most h . If $\mathcal{F} = (V, E)$ is a down-closed hypergraph with $N \geq 4n$ vertices and more than $(1 - (4d)^{-h}) \binom{N}{h}$ edges of cardinality h , then there are at least $(N/2)^n$ labeled copies of \mathcal{H} in \mathcal{F} .*

Proof. Call a subset $S \subset V$ of size $|S| \leq h$ *good* if S is contained in more than $(1 - (4d)^{|S|-h}) \binom{N}{h-|S|}$ edges of \mathcal{F} of cardinality h . For a good set S with $|S| < h$ and a vertex $j \in V \setminus S$, call j *bad* with respect to S if $S \cup \{j\}$ is not good. Let B_S denote the set of vertices $j \in V \setminus S$ that are bad with respect to S . The key observation is that if S is good with $|S| < h$, then $|B_S| \leq N/(4d)$. Indeed, suppose $|B_S| > N/(4d)$, then the number of h -sets containing S that are not edges of G is at least

$$\frac{|B_S|}{h - |S|} (4d)^{|S|+1-h} \binom{N}{h - |S| - 1} > (4d)^{|S|-h} \binom{N}{h - |S|},$$

which contradicts the fact that S is good.

Fix a labeling $\{v_1, \dots, v_n\}$ of the vertices of \mathcal{H} . Since the maximum degree of \mathcal{H} is d , for every vertex v_i there are at most d subsets $S \subset L_i = \{v_1, \dots, v_i\}$ containing v_i such that $S = e \cap L_i$ for some edge e of \mathcal{H} . We use induction on i to find many embeddings f of \mathcal{H} in \mathcal{F} such that for each edge e of \mathcal{H} , the set $f(e \cap L_i)$ is good.

By our definition, the empty set is good. Assume at step i , for all edges the sets $f(e \cap L_i)$ are good. There are at most d subsets S of L_{i+1} that are of the form $S = e \cap L_{i+1}$ where e is an edge of \mathcal{H} containing v_{i+1} . By the induction hypothesis, for each such subset S , the set $f(S \setminus \{v_{i+1}\})$ is good and therefore there are at most $\frac{N}{4d}$ bad vertices in \mathcal{F} with respect to it. In total this gives at most $d \frac{N}{4d} = N/4$ vertices. The remaining at least $3N/4 - i$ vertices in $\mathcal{F} \setminus f(L_i)$ are good with respect to all the above sets $f(S \setminus \{v_{i+1}\})$ and we can pick any of them to be $f(v_{i+1})$. Notice that this construction guarantees that $f(e \cap L_{i+1})$ is good for every edge e in \mathcal{H} . In the end of the process we obtain a mapping f such that $f(e \cap L_n) = f(e)$ is good for every e in \mathcal{H} . In particular, $f(e)$ is contained in at least one edge of \mathcal{F} of cardinality h and therefore $f(e)$ itself is an edge of \mathcal{F} since \mathcal{F} is down-closed. This shows that f is indeed an embedding of \mathcal{H} in \mathcal{F} . Since at step i we have at least $3N/4 - i$ choices for vertex v_{i+1} and since $N \geq 4n$, we get at least $\prod_{i=0}^{n-1} (\frac{3}{4}N - i) \geq (N/2)^n$ labeled copies of \mathcal{H} . \square

Using these two lemmas we can now complete the proof of Theorem 1.6, which implies also Theorem 1.1 and Corollaries 1.2, 1.3. For a graph G and a subset A , we let $G[A]$ denote the subgraph of G induced by A . If $G = (V, E)$ is a graph with N vertices and $\epsilon \binom{N}{2}$ edges, then, by averaging over all partitions $V = V_1 \cup V_2$ with $|V_1| = |V_2| = N/2$, we can find a partition with at least $\epsilon(N/2)^2$ edges between V_1 and V_2 . Hence, Theorem 1.6 follows from the following statement.

Theorem 2.3 *Let H be a bipartite graph with parts U_1 and U_2 , n vertices and maximum degree at most $d \geq 2$. If $\epsilon > 0$ and $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = N \geq 16d\epsilon^{-d}n$ and at least ϵN^2 edges, then G contains at least $(32d)^{-n/2} \epsilon^{dn} N^n$ labeled copies of H .*

Proof. Assume without loss of generality that $|U_2| \geq |U_1|$. Let \mathcal{H} be the hypergraph with vertex set U_2 such that a subset $D \subset U_2$ is an edge of \mathcal{H} if and only if there is a vertex $u \in U_1$ with $N_H(u) = D$. This \mathcal{H} has $|U_2| \leq n$ vertices, maximum degree at most d and edges of size at most d .

Let $x = \frac{\epsilon^d}{8d}N$, so in particular, $x \geq 2n \geq 4|U_1|$. We show that G contains many copies of H so that the vertices of U_i are embedded in V_i for $i \in \{1, 2\}$. Call a d -set $S \subset V_2$ *nice* if $|N_G(S)| \geq x$. Let \mathcal{F} be the down-closed hypergraph with vertex set V_2 whose edges are all subsets of V_2 which are contained in a nice d -set. An important observation is that each copy of \mathcal{H} in \mathcal{F} can be used to embed many distinct copies of H in G as follows. Suppose that $f : U_2 \rightarrow V_2$ is an embedding of \mathcal{H} in \mathcal{F} . For every copy of H use f to embed vertices in U_2 . Embed vertices in U_1 one by one. Suppose that the current vertex to embed is $u \in U_1$ and let D be the set of neighbors of u in U_2 . Then D is an edge of \mathcal{H} and therefore $f(D)$ is contained in a nice set and has at least x common neighbors in G . Since only at most $|U_1|$ of them can be occupied by other vertices of the copy of H which we are embedding, we still have at least $x - |U_1| \geq \frac{3}{4}x$ available vertices to embed u . Since this holds for every vertex in U_1 , altogether we get at least $(\frac{3}{4}x)^{|U_1|}$ distinct embeddings of H for each copy of \mathcal{H} in \mathcal{F} .

Next we will find a large induced subhypergraph of \mathcal{F} which is sufficiently dense to apply Lemma 2.2. By Lemma 2.1 with $a = t = d$, V_2 contains a subset A of size $|A| \geq 2^{-1/d} \epsilon^d N \geq 2^{-1/2} \epsilon^d N$ such that the number of d -sets $S \subset A$ satisfying $|N_G(S)| < x$ is at most

$$\begin{aligned} 2\epsilon^{-d^2} \left(\frac{x}{N}\right)^d \binom{|A|}{d} \binom{N}{d} &= 2\epsilon^{-d^2} \left(\frac{\epsilon^d N / 8d}{N}\right)^d \left(\frac{|A|}{N}\right)^d \binom{N}{d} \\ &= 2(8d)^{-d} \left(\frac{|A|}{N}\right)^d \binom{N}{d} \\ &< (4d)^{-d} \binom{|A|}{d}. \end{aligned}$$

Here we use that $|A|^d \leq 2^{d-1} d! \binom{|A|}{d}$ which follows from $d \geq 2$ and $|A| \geq 2^{-1/d} \epsilon^d N > 8d$.

Applying Lemma 2.2 with $h = d$, to the subhypergraph $\mathcal{F}[A]$ induced by the set A , we obtain at least $\left(\frac{|A|}{2}\right)^{|U_2|}$ labeled copies of \mathcal{H} . By the above discussion each such copy of \mathcal{H} can be extended to $(\frac{3}{4}x)^{|U_1|}$ labeled copies of H . Therefore, using that $|U_1| \leq |U_2|$, $|U_1| + |U_2| = n$, $|A| \geq 2^{-1/2} \epsilon^d N$ and $x = \frac{\epsilon^d}{8d}N$, we conclude that G contains at least

$$\left(\frac{|A|}{2}\right)^{|U_2|} \left(\frac{3}{4}x\right)^{|U_1|} \geq \left(\frac{3}{32}\right)^{-|U_1|} (2^{-3/2})^{-|U_2|} d^{-n/2} \epsilon^{dn} N^n \geq (32d)^{-n/2} \epsilon^{dn} N^n$$

labeled copies of H , completing the proof. \square

3 Degenerate bipartite graphs

The main result of this section is the following theorem which implies Theorem 1.4.

Theorem 3.1 *Let H be a d -degenerate bipartite graph with n vertices and maximum degree Δ . Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = N$ vertices and at least ϵN^2 edges. Suppose $d \geq 2, d/n \leq \delta \leq 1$ and let $x = 2^{-9} \epsilon^{(1+(1+\delta^{-1})d)(1+\delta)} \Delta^{-\delta} N$. If $x \geq 4n$, then G contains at least $(x/4)^n$ labeled copies of H .*

To obtain from this statement Theorem 1.4, recall that every graph with N vertices and $\epsilon \binom{N}{2}$ edges, has a partition $V = V_1 \cup V_2$ with $|V_1| = |V_2| = N/2$ such that the number of edges between V_1 and V_2 is at least $\epsilon(N/2)^2$. Moreover, our result shows that if H is a bipartite d -degenerate graph of order n and maximum degree at most exponential in d , then every large graph G with edge density ϵ contains at least a fraction $\epsilon^{O(dn)}$ of all possible copies of H . This is best possible up to the constant factor in the exponent and shows that Sidorenko's conjecture discussed in Section 1.2 is not very far from being true.

Proof of Theorem 3.1. Let $t = (1 + \delta^{-1})d$ and $u = t + d$. By Lemma 2.1 with parameters $a = 1, u, t, x$, V_1 contains a subset A' with $|A'| \geq \frac{1}{2} \epsilon^t N$ such that the number Y of u -sets $T \subset A'$ with $|N(T)| < x$ is at most

$$Y \leq 2\epsilon^{-t} \left(\frac{x}{N}\right)^t \left(\frac{|A'|}{N}\right) \binom{N}{u} \leq 2\epsilon^{-t} \left(\frac{x}{N}\right)^t \binom{N}{u}.$$

Let S be a random subset of A' of size t and let $A_2 = N(S)$. Denote by Q the random variable counting the number of u -sets $T \subset A'$ containing S such that $|N(T)| < x$. Note that the number of u -sets T with $|N(T)| < x$ is at most Y and each of them contains the random subset S with probability $\binom{u}{t} / \binom{|A'|}{t}$. Thus, using that $t - d = \delta^{-1}d$, $u = t + d$, $\frac{u^u}{u!} < e^u < 2^{3u/2}$ and $|A'| \geq 2x \geq u$, we have

$$\begin{aligned} \mathbb{E}[Q] &= \frac{\binom{u}{t}}{\binom{|A'|}{t}} Y \leq \frac{\binom{u}{t}}{\binom{|A'|}{t}} 2\epsilon^{-t} \left(\frac{x}{N}\right)^t \binom{N}{u} \leq 2 \left(\frac{ux}{\epsilon |A'| N}\right)^t \binom{N}{u} \\ &\leq 2 \left(\frac{ux}{\frac{1}{2} \epsilon^{t+1} N^2}\right)^t \binom{N}{u} \leq 2^{t+1} \frac{u^t}{u!} \epsilon^{-(t+1)t} \left(\frac{x}{N}\right)^{t-d} x^d \\ &= 2^{t+1} \frac{u^u}{u!} 2^{-9d/\delta} \Delta^{-d} \frac{x^d}{u^d} < \frac{1}{2} (2\Delta)^{-d} \binom{x}{d}. \end{aligned}$$

It is important to observe that Q also gives an upper bound on the number of d -sets T' in $A' \setminus S$ which have less than x common neighbors in A_2 . Indeed, we can correspond to every such T' a set $T = T' \cup S$. Since $N(T) = N(S) \cap N(T') = A_2 \cap N(T')$, T has less than x common neighbors. Therefore the number of sets T' is bounded by the number of sets T . Let $A_1 = A' \setminus S$. Then, using that $t = d + \delta^{-1}d \leq 2n \leq x/2$, we have that $|A_1| = |A'| - |S| \geq 2x - t \geq x$.

Let Z denote the random variable counting the number of subsets of A_2 with cardinality d with less than x common neighbors in A_1 . Note that such a set has at most $t + x \leq 2x$ common neighbors in A' . For a given d -set $R \subset V_2$, the probability that R is a subset of A_2 is $\binom{|N_{A'}(R)|}{t} \binom{|A'|}{t}^{-1} \leq \left(\frac{|N_{A'}(R)|}{|A'|}\right)^t$.

Therefore, using that $t = d + \delta^{-1}d$, we have

$$\begin{aligned}\mathbb{E}[Z] &< \binom{N}{d} \left(\frac{2x}{|A'|} \right)^t < \frac{N^d}{d!} \left(\frac{2x}{\frac{1}{2}\epsilon^t N} \right)^t = 2^{2t} \epsilon^{-t^2} \left(\frac{x}{N} \right)^{t-d} \frac{x^d}{d!} \\ &= 2^{2t-9d/\delta} \epsilon^t \Delta^{-d} \frac{x^d}{d!} < \frac{1}{2} (2\Delta)^{-d} \binom{x}{d}.\end{aligned}$$

Since Q and Z are nonnegative discrete random variables, by Markov's inequality, $\mathbb{P}[Q > 2\mathbb{E}[Q]] < 1/2$ and $\mathbb{P}[Z > 2\mathbb{E}[Z]] < 1/2$. Thus there is a choice of set S such that

$$Q \leq 2\mathbb{E}[Q] < (2\Delta)^{-d} \binom{x}{d}$$

and

$$Z \leq 2\mathbb{E}[Z] < (2\Delta)^{-d} \binom{x}{d}.$$

Since $Q < \binom{x}{d}$ and $|A_1| \geq x$, then there is a d -set in A_1 that has at least x common neighbors in A_2 and so $|A_2| \geq x$. Therefore, for each $i \in \{1, 2\}$, $|A_i| \geq x$ and all but less than $(2\Delta)^{-d} \binom{x}{d}$ subsets of A_i of size d have at least x common neighbors in A_{3-i} . By Lemma 3.2 applied to the induced subgraph of G by $A_1 \cup A_2$, we have that G contains at least $(x/4)^n$ labeled copies of H . \square

Lemma 3.2 *Let $H = (U_1, U_2; F)$ be a d -degenerate bipartite graph with n vertices and maximum degree Δ . Let $G = (A_1, A_2; E)$ be a bipartite graph such that for $i \in \{1, 2\}$, $|A_i| \geq x \geq 4n$ and the number of d -sets $U \subset A_i$ with $N(U) < x$ is less than $(2\Delta)^{-d} \binom{x}{d}$. Then G contains at least $(x/4)^n$ labeled copies of H .*

Proof. A d -set $S \subset A_i$ is *good* if $|N(S)| \geq x$, otherwise it is *bad*. Also, a subset $U \subset A_i$ with $|U| < d$ is *good* if it is contained in less than $(2\Delta)^{|U|-d} \binom{x}{d-|U|}$ bad subsets of A_i of size d . A vertex $v \in A_i$ is *bad with respect to a subset* $U \subset A_i$ with $|U| < d$ if U is good but $U \cup \{v\}$ is not. Note that, for any good subset $U \subset A_i$ with $|U| < d$, there are at most $\frac{x}{2\Delta}$ vertices that are bad with respect to U . Indeed, if not, then there would be more than

$$\frac{x/(2\Delta)}{d-|U|} (2\Delta)^{|U|+1-d} \binom{x}{d-|U|-1} \geq (2\Delta)^{|U|-d} \binom{x}{d-|U|}$$

subsets of A_i of size d containing U that are bad, which would contradict U being good.

Since H is d -degenerate, then there is an ordering $\{v_1, \dots, v_n\}$ of the vertices of H such that each vertex v_i has at most d neighbors v_j with $j < i$. Let $N^-(v_i)$ be all the neighbors v_j of v_i with $j < i$. Let $L_h = \{v_1, \dots, v_h\}$. We will use induction on h to find at least $(x/4)^n$ embeddings f of H in G such that $f(U_i) \subset A_i$ for $i \in \{1, 2\}$ and for every vertex v_j and every $h \in [n]$, the set $f(N^-(v_j) \cap L_h)$ is good.

By our definition, the empty set is good for each $i \in \{1, 2\}$. We will embed the vertices in the increasing order of their indices. Suppose we are embedding v_h . Then, by the induction hypothesis, for each vertex v_j , the set $f(N^-(v_j) \cap L_{h-1})$ is good. Since the set $f(N^-(v_h \cap L_{h-1})) = f(N^-(v_h))$ is good, it has at least x common neighbors. Also, v_h has degree at most Δ , so there are at most Δ sets $f(N^-(v_j) \cap L_{h-1})$ where v_j is a neighbor of v_h and $j > h$. These sets are good, so there are

at most $\Delta \frac{x}{2\Delta} = x/2$ vertices which are bad for at least one of them. This implies that there at least $x - x/2 - (h - 1) > x/4$ vertices in the common neighborhood of $f(N^-(v_h))$ which are not occupied yet and are good for all the above sets $f(N_-(v_j) \cap L_{h-1})$. Any of these vertices can be chosen as $f(v_h)$. Altogether, we get at least $(x/4)^n$ labeled copies of H . \square

This proof can be modified to obtain the bound $r(H) \leq 2^c n$ for p -arrangeable bipartite H , where c is some absolute constant. Note that the maximum degree Δ of H is only used in the last paragraph to bound the number of sets $f(N_-(v_j) \cap L_{h-1})$ where v_j is a neighbor of v_h and $j > h$. As we already discussed in detail in the introduction if graph H is p -arrangeable then there is an ordering of its vertices for which the number of distinct sets $N_-(v_j) \cap L_{h-1}$ where v_j is a neighbor of v_h and $j > h$ is bounded by 2^{p-1} for every h . Therefore, we can use for p -arrangeable bipartite graphs the same proof as for p -degenerate bipartite graphs with maximum degree at most 2^{p-1} . We easily obtain the following slight variant of Lemma 3.2 for the proof.

Lemma 3.3 *Let $H = (U_1, U_2; F)$ be a p -arrangeable bipartite graph with n vertices. Let $G = (A_1, A_2; E)$ be a bipartite graph such that for $i \in \{1, 2\}$, $|A_i| \geq x \geq 4n$ and the number of p -sets $U \subset A_i$ with $N_G(U) < x$ is less than $2^{-p^2} \binom{x}{p}$. Then G contains at least $(x/4)^n$ labeled copies of H .*

The remaining details of the proof are essentially identical and therefore omitted.

4 Graphs with bounded chromatic number

The following result implies Theorem 1.5 since every graph with chromatic number q and maximum degree d satisfies $q \leq d + 1$ and hence $(2d + 2)(2q - 3) + 2 \leq 4dq$. Moreover, Theorem 4.1 shows that every 2-edge-coloring of K_N with $N \geq 2^{4dq}n$ contains at least $2^{-4dq}N^n$ labeled monochromatic copies of any n -vertex graph H with chromatic number q and maximum degree d . This implies that the Ramsey multiplicity for graphs with fixed chromatic number and whose average degree is at least a constant fraction of the maximum degree is not very far from the bound given by a random coloring.

Theorem 4.1 *If H is a graph with n vertices, chromatic number q , and maximum degree $d \geq 2$, then for every 2-edge-coloring of K_N with $N \geq 2^{(2d+2)(2q-3)+2}n$, there are at least $(2^{-(2d+2)(2q-3)-2}N)^n$ labeled monochromatic copies of H .*

Proof. Consider a 2-edge-coloring of K_N with colors 0 and 1. For $j \in \{0, 1\}$, let G_j denote the graph of color j . Let A_1 be the vertex set of K_N and $x = 2^{-(2d+2)(2q-3)}N$, so $x \geq 4n$. We will pick subsets $A_1 \supset A_2 \supset \dots \supset A_{2q-2}$ such that for each $i \leq 2q - 3$, we have $|A_{i+1}| \geq |A_i|/2^{2d+2}$ and there is a color $c(i) \in \{0, 1\}$ such that there are less than $(2d)^{-d} \binom{x}{d}$ d -sets $U \subset A_{i+1}$ which have less than x common neighbors in the induced subgraph $G_{c(i)}[A_i]$.

Given A_i , we can pick $c(i)$ and A_{i+1} as follows. Arbitrarily partition A_i into two subsets $A_{i,1}$ and $A_{i,2}$ of equal size. Let $c(i)$ denote the densest of the two colors between $A_{i,1}$ and $A_{i,2}$. By Lemma 2.1 with $\epsilon = 1/2$, $a = 1$, and $t = 2d$, there is a subset $A_{i+1} \subset A_{i,2} \subset A_i$ with $|A_{i+1}| \geq 2^{-2d-1}|A_{i,2}| = 2^{-2d-2}|A_i|$ such that for all but at most

$$2 \cdot 2^{2d} \left(\frac{x}{|A_i|} \right)^{2d} \left(\frac{|A_{i+1}|}{|A_{i,2}|} \right) \binom{|A_{i,2}|}{d} \leq 2^{2d+1} \left(\frac{x}{|A_i|} \right)^{2d} \left(\frac{|A_i|/2}{d} \right) < 2^{d+1} \left(\frac{x}{|A_i|} \right)^d \frac{x^d}{d!} < (2d)^{-d} \binom{x}{d}$$

d -sets $U \subset A_{i+1}$, U has at least x common neighbors in $G_{c(i)}[A_i]$. Here, the last inequality uses the fact that $|A_i| \geq 2^{-(i-1)(2d+2)}N \geq 2^{-(2q-4)(2d+2)}N = 2^{2d+2}x$.

Given the subsets $A_1 \supset \dots \supset A_{2q-2}$ with the desired properties and the colors $c(1), \dots, c(2q-3)$, notice that $|A_{2q-2}| \geq 2^{-(2d+2)(2q-3)}N = x$. By the pigeonhole principle, one of the two colors is represented at least $q-1$ times in the sequence $c(1), \dots, c(2q-3)$. Without loss of generality suppose that 0 is this popular color. Let $V_1 = A_1$, and for $1 \leq k < q$, let $V_{k+1} = A_{j+1}$, where j is the k^{th} smallest positive integer such that $c(j) = 0$. By applying Lemma 4.2 below to the graph G_0 and subsets V_1, \dots, V_q , we can find at least $(x/4)^n$ labeled monochromatic copies of H , which completes the proof. \square

Lemma 4.2 *Suppose G is a graph with vertex set V_1 , and let $V_1 \supset \dots \supset V_q$ be a family of nested subsets of V_1 such that $|V_q| \geq x \geq 4n$, and for $1 \leq i < q$, all but less than $(2d)^{-d} \binom{x}{d}$ d -sets $U \subset V_{i+1}$ satisfy $|N(U) \cap V_i| \geq x$. Then, for every q -partite graph H with n vertices and maximum degree at most d , there are at least $(x/4)^n$ labeled copies of H in G .*

Proof. A d -set $S \subset V_{i+1}$ is *good with respect to i* if $|N(S) \cap V_i| \geq x$, otherwise it is *bad with respect to i* . Also, a subset $U \subset V_{i+1}$ with $|U| < d$ is *good with respect to i* if there are less than $(2d)^{|U|-d} \binom{x}{d-|U|}$ subsets of V_{i+1} of size d that contain U and are bad with respect to i . For a good subset $U \subset V_{i+1}$ with respect to i with $|U| < d$, call a vertex $v \in V_{i+1}$ *bad with respect to U and i* if $U \cup \{v\}$ is bad with respect to i . For any i and subset $U \subset V_{i+1}$ that is good with respect to i , there are less than $\frac{x}{2d}$ bad vertices with respect to U and i . Indeed, if otherwise, then the number of subsets of V_{i+1} of size d containing U that are bad is at least

$$\frac{x/(2d)}{d-|U|} (2d)^{|U|+1-d} \binom{x}{d-|U|-1} \geq (2d)^{|U|-d} \binom{x}{d-|U|},$$

which contradicts the fact that U is good with respect to i .

Consider a partition $W_1 \cup \dots \cup W_q$ of the vertices of H into q independent sets. Order the vertices $\{v_1, \dots, v_n\}$ of H such that the vertices of W_i precede the vertices of W_j for $i > j$. Let $L_h = \{v_1, \dots, v_h\}$. For a vertex v_j , let $N^-(v_j)$ denote the set of vertices $v_i, i < j$ adjacent to v_j and $N^+(v_j)$ denote the set of vertices $v_i, i > j$ adjacent to v_j . By our ordering of the vertices of H and the fact that each W_k is an independent set, if $w \in W_k, v \in N^-(w)$, and $v \in W_\ell$, then $\ell > k$. Similarly, if $w \in W_k, v \in N^+(w)$, and $v \in W_\ell$, then $\ell < k$.

We use induction on h to find many embeddings f of H in G such that $f(W_k) \subset V_k$ for all k , and the set $f(L_h \cap N^-(w))$ is *good* with respect to k for all h, k , and $w \in W_k$. Since $f(W_i) \subset V_i$ and the sets V_i are nested, by the above discussion we also have that $f(N^-(w)) \subset V_{k+1}$ for all $w \in W_k$. By our definition, the empty set is good with respect to every k , which demonstrates the base case $h = 0$ of the induction. We pick the vertices for the embedding in order of their index. Suppose we are embedding v_h with $v_h \in W_\ell$. Our induction hypothesis is that we have already embedded L_{h-1} with the desired properties, so for each k and $w \in W_k$, the set $f(L_{h-1} \cap N^-(w)) \subset V_{k+1}$ is good with respect to k . We need to show how to pick $f(v_h) \in V_\ell$ that is not already occupied such that $f(v_h)$ is adjacent to all vertices in $f(N^-(v_h))$ and for each vertex $w \in N^+(v_h)$ with $w \in W_j$, $f(v_h)$ is good with respect to $f(N^-(w) \cap L_{h-1})$ and j .

Since $f(N^-(v_h) \cap L_{h-1}) = f(N^-(v_h))$ is good with respect to ℓ , then $f(N^-(v_h))$ is contained in a d -set that is good with respect to ℓ and so it has at least x common neighbors in V_ℓ . Let $w \in N^+(v_h)$

such that $w \in W_j$, then $j < \ell$. Since $V_\ell \subset V_{j+1}$, then there are less than $\frac{x}{2d}$ vertices in V_ℓ that are bad with respect to $f(N^-(w) \cap L_{h-1})$ and j . Since there are at most d such w , then there are at least $x - d\frac{x}{2d} - (h-1) \geq x/4$ unoccupied vertices in V_ℓ satisfying the above properties, which we can choose for $f(v_h)$. Altogether, we get at least $(x/4)^n$ copies of H in G . \square

The constant factor in the exponent in Theorems 4.1 and 1.5 can be improved for large q by roughly a factor of 2 by picking $t \approx d + \log d$ instead of $t = 2d$. Also, the above proof can be easily adapted to give the following upper bound on multicolor Ramsey numbers.

Theorem 4.3 *If H_1, \dots, H_k are $k \geq 2$ graphs with at most n vertices, chromatic number at most q , and maximum degree at most Δ , then*

$$r(H_1, \dots, H_k) \leq k^{2k\Delta q} n.$$

5 Density theorem for subdivided graphs

Note that the 1-subdivision of a graph Γ is a bipartite graph whose first part contains the vertices of Γ and whose second part contains the vertices which were used to subdivide the edges of Γ . Furthermore, the vertices in the second part have degree two. Also, if Γ has n edges and no isolated vertices then its 1-subdivision has at most $3n$ vertices. Therefore, Theorem 1.7 follows from the following theorem.

Theorem 5.1 *If $H = (U_1, U_2; F)$ is a bipartite graph with n vertices such that every vertex in U_2 has degree 2, G is a graph with $2N$ vertices, $2\epsilon N^2$ edges, and $N \geq 128\epsilon^{-3}n$, then H is a subgraph of G .*

Proof. By averaging over all partitions $V = V_1 \cup V_2$ of G with $|V_1| = |V_2| = N$, we can find a partition with at least ϵN^2 edges between V_1 and V_2 . Delete the vertices of V_1 with less than $\epsilon N/2$ neighbors in V_2 , and let V'_1 denote the set of remaining vertices of V_1 . Note that we deleted at most $\epsilon N^2/2$ edges so between V'_1 and V_2 there are still at least $\epsilon N^2/2$ edges. Let G' be the graph with parts V'_1, V_2 and all edges between them. Every vertex in V'_1 has degree at least $\epsilon N/2$ in G' and $|V'_1| \geq \epsilon N/2$.

Let H' be the graph with vertex set U_1 such that two vertices in U_1 are adjacent in H' if and only if they have a neighbor in common. Since $|U_2| + |U_1| = n$, then the number of edges of H' is at most n . Consider an auxiliary graph G^* with vertex set V'_1 such that two vertices of V'_1 are adjacent if their common neighborhood in G' has cardinality at least n . Note that given an embedding $f : U_1 \rightarrow V_1$ of H' in G^* , we can extend it to an embedding of H in G' as follows. Use f to embed vertices in U_1 . Embed vertices in U_2 one by one. Suppose that the current vertex to embed is $u \in U_2$ and let D be the set of neighbors of u in U_1 , so $|D| = 2$. Then D is an edge in H' and so $f(D)$ is an edge of G^* . Therefore, $f(D)$ has at least n common neighbors in G' . As the total number of vertices of H embedded so far is less than n , one of the common neighbors of $f(D)$ is still unoccupied and can be used to embed u . Thus it is enough to find a copy of H' in G^* .

To do this, we construct a family of nested subsets $V'_1 = A_0 \supset A_1 \supset \dots \supset \dots$ such that for all $i \geq 1$, $|A_i| \geq \frac{\epsilon}{8}|A_{i-1}|$ and the maximum degree in the complement of the induced subgraph $G^*[A_i]$ is at most $(\epsilon/8)^i|A_i|$. Set $c_i = (\epsilon/8)^i$ and let E_i be the set of edges of $\bar{G}^*[A_i]$. Then $|E_i| \leq c_i|A_i|^2/2$.

Having already picked A_1, \dots, A_{i-1} satisfying the above two desired properties, we show how to pick A_i . Let w be a vertex from V_2 chosen uniformly at random. Let A denote the intersection of

A_{i-1} with the neighborhood of w , and X be the random variable denoting the cardinality of A . Since every vertex in V_1' has degree at least $\epsilon N/2$,

$$\mathbb{E}[X] = \sum_{v \in A_{i-1}} \frac{|N_{G'}(v)|}{|V_2|} \geq \frac{\epsilon}{2} |A_{i-1}|.$$

Let Y be the random variable counting the number of pairs in A with fewer than n common neighbors in V_2 , i.e., Y counts the number of pairs in A that are not edges of G^* . Notice that the probability that a pair R of vertices of A_{i-1} is in A is at most $\frac{|N_{G'}(R)|}{|V_2|}$. Recall that E_{i-1} is the set of all pairs R in A_{i-1} with $|N_{G'}(R)| < n$ (these are edges of \bar{G}^*) and $|E_{i-1}| \leq c_{i-1} |A_{i-1}|^2/2$. Therefore, we have

$$\mathbb{E}[Y] < \frac{n}{N} |E_{i-1}| \leq \frac{n}{N} \frac{c_{i-1}}{2} |A_{i-1}|^2.$$

By convexity, $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$. Thus, using linearity of expectation, we obtain

$$\mathbb{E} \left[X^2 - \frac{\mathbb{E}[X]^2}{2\mathbb{E}[Y]} Y - \mathbb{E}[X]^2/2 \right] \geq 0.$$

Therefore, there is a choice of w such that this expression is nonnegative. Then

$$X^2 \geq \frac{1}{2} \mathbb{E}[X]^2 \geq \frac{\epsilon^2}{8} |A_{i-1}|^2$$

and

$$Y \leq 2 \frac{X^2}{\mathbb{E}[X]^2} \mathbb{E}[Y] \leq 4\epsilon^{-2} c_{i-1} \frac{n}{N} X^2 \leq \frac{\epsilon}{16} c_{i-1} \frac{X^2}{2}.$$

From the first inequality, we have $|A| = X \geq \frac{\epsilon}{4} |A_{i-1}|$ and the second inequality implies that the average degree in the induced subgraph $\bar{G}^*[A]$ is at most $\epsilon c_{i-1} |A|/16$. If A contains a vertex of degree more than $\epsilon c_{i-1} |A|/16$, then delete it, and continue this process until the remaining induced subgraph of $\bar{G}^*[A]$ has maximum degree at most $\epsilon c_{i-1} |A|/16$. Let A_i denote the vertex set of this remaining induced subgraph. Clearly, the number of deleted edges is at least $(|A| - |A_i|) \epsilon c_{i-1} |A|/16$. As explained above, the number of edges of $G^*[A]$ is at most $\epsilon c_{i-1} |A|^2/32$, so we arrive at the inequality $|A| - |A_i| \leq |A|/2$. Hence, $|A_i| \geq |A|/2 \geq \frac{\epsilon}{8} |A_{i-1}|$ and the maximum degree in $\bar{G}^*[A_i]$ is at most $\frac{\epsilon}{16} c_{i-1} |A| \leq \frac{\epsilon}{8} c_{i-1} |A_i| = c_i |A_i|$. Therefore, we have shown how to find the nested family of subsets with the desired properties.

Label the vertices $\{v_1, \dots, v_{|U_1|}\}$ of H' in decreasing order of their degree. Since H' has at most n edges, the degree of v_i is at most $2n/i$. We will find an embedding f of H' in G^* which embeds vertices in the order of their index i . The vertex v_i will be embedded in A_j where j is the least positive integer such that $c_j \leq \frac{i}{4n}$. Since $c_j = (\frac{\epsilon}{8})^j$, then

$$|A_j| \geq c_j |A_0| \geq c_j \frac{\epsilon N}{2} \geq \frac{\epsilon}{8} \frac{i}{4n} \frac{\epsilon N}{2} \geq 2i.$$

Assume we have already embedded the vertices $\{v_k; k < i\}$ and we want to embed v_i . Let $N^-(v_i)$ be the set of vertices $v_k, k < i$ that are adjacent to v_i in H' . The maximum degree in the induced subgraph $\bar{G}^*[A_j]$ is at most $c_j |A_j| \leq \frac{i}{4n} |A_j|$. Since v_i has degree at most $\frac{2n}{i}$ in H' , then at least $|A_j| - \frac{2n}{i} \cdot \frac{i}{4n} |A_j| \geq |A_j|/2$ vertices of A_j are adjacent in G^* to all the vertices in $f(N^-(v_i))$. Since also $|A_j|/2 \geq i$, then there is a vertex in $A_j \setminus f(\{v_1, \dots, v_{i-1}\})$ that is adjacent in G^* to all the vertices of $f(N^-(v_i))$. Use this vertex to embed v_i and continue. This gives a copy of H' in G^* , completing the proof. \square

6 Embedding induced subgraphs

To prove the results stated in Sections 1.4 - 1.6, we need the following embedding lemma for induced subgraphs.

Lemma 6.1 *Let G and F be two edge-disjoint graphs on the same vertex set U and let $A_1 \supset \dots \supset A_n$ be vertex subsets of U with $|A_n| \geq m \geq 2n$ for some positive integers m and n . Suppose that for every $i < n$, all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs (S_1, S_2) of disjoint subsets of A_{i+1} with $|S_1| = |S_2| = n$ have at least m vertices in A_i that are adjacent to S_1 in G and are adjacent to S_2 in F . Then, for each graph H with n vertices $V = \{v_1, \dots, v_n\}$, there is an embedding $f : V \rightarrow U$ such that for every pair $i < j$, $(f(v_i), f(v_j))$ is an edge of G if (v_i, v_j) is an edge of H , and $(f(v_i), f(v_j))$ is an edge of F if (v_i, v_j) is not an edge of H .*

Proof. Call a pair (S_1, S_2) of disjoint subsets of A_{i+1} with $|S_1| = |S_2| = n$ *good with respect to i* if there are at least m vertices in A_i that are adjacent to S_1 in G and adjacent to S_2 in F , otherwise it is *bad with respect to i* . Also, call a pair (U_1, U_2) of disjoint subsets of A_{i+1} each of cardinality at most n *good with respect to i* if less than $(2n)^{|U_1|+|U_2|-2n} \binom{m}{n-|U_1|} \binom{m}{n-|U_2|}$ pairs (S_1, S_2) of disjoint subsets of A_{i+1} with $|S_1| = |S_2| = n$, $U_1 \subset S_1$, and $U_2 \subset S_2$ are bad with respect to i , otherwise it is *bad with respect to i* . Note that if (U_1, U_2) is good with respect to i , then there is a pair (S_1, S_2) of disjoint subsets of A_{i+1} with $|S_1| = |S_2| = n$, $U_1 \subset S_1$ and $U_2 \subset S_2$ that is good with respect to i , so

$$|N_G(U_1) \cap N_F(U_2) \cap A_i| \geq |N_G(S_1) \cap N_F(S_2) \cap A_i| \geq m.$$

For $b \in \{1, 2\}$ and a pair (U_1, U_2) of subsets of A_{i+1} that is good with respect to i with $|U_b| < n$, call a vertex $w \in A_{i+1}$ *bad with respect to (U_1, U_2, b, i)* if $b = 1$ and $(U_1 \cup \{w\}, U_2)$ is bad with respect to i , or if $b = 2$ and $(U_1, U_2 \cup \{w\})$ is bad with respect to i . For $b \in \{1, 2\}$ and a pair (U_1, U_2) of subsets of A_{i+1} that is good with respect to i , there are less than $\frac{m}{2n}$ vertices $w \in A_{i+1}$ that are bad with respect to (U_1, U_2, b, i) . Indeed, otherwise the number of pairs (S_1, S_2) of subsets of A_{i+1} each of size n with $U_1 \subset S_1$ and $U_2 \subset S_2$ that are bad with respect to i is at least

$$\frac{m/(2n)}{n - |U_b|} (2n)^{|U_{3-b}|+|U_b|+1-2n} \binom{m}{n - |U_b| - 1} \binom{m}{n - |U_{3-b}|} \geq (2n)^{|U_1|+|U_2|-2n} \binom{m}{n - |U_1|} \binom{m}{n - |U_2|},$$

which contradicts the fact that (U_1, U_2) is good with respect to i .

We next show how to find a copy of H in G such that vertex pairs in this copy corresponding to nonedges of H are edges of F . We embed the vertices of H one by one in the increasing order of their index. Let $L_h = \{v_1, \dots, v_h\}$. For a vertex v_j , let $N^-(v_j)$ denote the vertices v_i adjacent to v_j with $i < j$ and $N^+(v_j)$ denote the vertices v_i adjacent to v_j with $i > j$. We use induction on h to construct the embedding f of H such that $f(v_j) \in A_{n-j+1}$ for all j , and for every v_j and $h < j$, the pair $(f(L_h \cap N^-(v_j)), f(L_h \setminus N^-(v_j)))$ is good with respect to $n - j + 1$.

The induction hypothesis is that we have already embedded L_{h-1} and for every v_j , the pair $(f(L_{h-1} \cap N^-(v_j)), f(L_{h-1} \setminus N^-(v_j)))$ is good with respect to $n - j + 1$. In the base case $h = 1$, the induction hypothesis holds since our definition implies that the pair (\emptyset, \emptyset) is good with respect to j for every j . Since the sets are nested, we have $f(L_{h-1}) \subset A_k$ for any $k \leq n - h + 2$. We need to show how to pick $f(v_h) \in V_{n-h+1}$ that is not already occupied and satisfies

- $f(v_h)$ is adjacent to $f(N^-(v_h))$ in G and adjacent to $f(L_{h-1} \setminus N^-(v_h))$ in F ,

- for each vertex $v_j \in N^+(v_h)$, $f(v_h)$ is not bad with respect to

$$(f(N^-(v_j) \cap L_{h-1}), f(L_{h-1} \setminus N^-(v_j)), 1, n - j + 1), \text{ and}$$

- for each vertex $v_j \notin N^+(v_h)$ with $j > h$, $f(v_h)$ is not bad with respect to

$$(f(N^-(v_j) \cap L_{h-1}), f(L_{h-1} \setminus N^-(v_j)), 2, n - j + 1).$$

Since $(L_{h-1} \cap f(N^-(v_h)), f(L_{h-1} \setminus N^-(v_h)))$ is good with respect to $n - h + 1$, then there are at least m vertices in A_{n-h+1} that are adjacent to every vertex of $f(N^-(v_h)) = f(L_{h-1} \cap N^-(v_h))$ in G and are adjacent to every vertex of $f(L_{h-1} \setminus N^-(v_h))$ in F . For each $v_j \in N^+(v_h)$, there are less than $\frac{m}{2n}$ vertices of A_{n-h+1} that are bad with respect to $(f(N^-(v_j) \cap L_{h-1}), f(L_{h-1} \setminus N^-(v_j)), 1, n - j + 1)$. Also, for each $v_j \notin N^+(v_h)$ with $j > h$, there are less than $\frac{m}{2n}$ vertices of A_{n-h+1} that are bad with respect to $(f(N^-(v_j) \cap L_{h-1}), f(L_{h-1} \setminus N^-(v_j)), 2, n - j + 1)$. Since the number of v_j with $j > h$ is $n - h$ and the number of already occupied vertices is $h - 1$, then there are at least

$$m - (n - h) \frac{m}{2n} - (h - 1) > m/2 - (n - 1) \geq 1$$

unoccupied vertices to choose for $f(v_h) \in A_{n-h+1}$ satisfying the above three desired properties, which, by induction on h , completes the proof. \square

A graph is n -universal if it contains all graphs on n vertices as induced subgraphs. For the proofs of Theorems 1.8 - 1.10 and Corollary 1.11, we need the special case $F = \bar{G}$ of the above lemma, which is stated below.

Corollary 6.2 *Let m and n be positive integers and let $A_1 \supset \dots \supset A_n$ be vertex subsets of a graph G with $|A_n| \geq m \geq 2n$. If for all $i < n$, all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs (S_1, S_2) of disjoint subsets of A_{i+1} with $|S_1| = |S_2| = n$ have at least m vertices in A_i that are adjacent to all vertices in S_1 and no vertices in S_2 , then graph G is n -universal.*

7 Ramsey-type results for H -free graphs

The purpose of this section is to prove Theorems 1.8 and 1.9 which are related to the Erdős-Hajnal conjecture. We first give an overview of the proofs before jumping into the details.

Lemma 7.1 below demonstrates that for a (large enough) graph G that is not too sparse and does not contain a pair of large subsets with edge density almost 1 between them, there is a large vertex subset A with the property that almost all pairs (S_1, S_2) of disjoint subsets of A of size n satisfy that $|N_G(S_1) \cap N_{\bar{G}}(S_2)|$ is large. The first step in the proof of Lemma 7.1 uses Lemma 2.1 to get a large subset A for which almost all vertex subsets S_1 of size n have large common neighborhood. Using the fact that G does not contain a pair of large subsets with edge density almost 1 between them, we show that for almost all pairs (S_1, S_2) of subsets of A of size n , $|N_G(S_1) \cap N_{\bar{G}}(S_2)|$ is large.

By repeated application of Lemma 7.1 and an application of Corollary 6.2, we arrive at Lemma 7.2, which says that every graph is n -universal, or contains a large independent set, or has two large subsets with edge density almost 1 between them. The deductions of Theorems 1.8 and 1.9 from Lemma 7.2 are relatively straightforward.

Lemma 7.1 *Suppose z is a positive integer, $\beta, \epsilon > 0$, and $G = (V, E)$ is a graph on N vertices and at least $\beta \binom{N}{2}$ edges such that for each pair (W_1, W_2) of disjoint subsets of V each of cardinality at least z , there is a vertex in W_1 with less than $(1 - 2\epsilon)|W_2|$ neighbors in W_2 . If $2 \leq n \leq z$ and m satisfy*

$$4nz^{1/2n}N^{1-1/2n} \leq m \leq \frac{\beta^2 \epsilon^{2n} N}{16n},$$

then there is a subset $A \subset V$ with $|A| \geq \frac{1}{4}\beta^{4n}N$ such that all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs (S_1, S_2) of disjoint subsets of A with $|S_1| = |S_2| = n$ have at least m vertices of G adjacent to every vertex in S_1 and no vertex in S_2 .

Proof. By averaging over all partitions $V = V_1 \cup V_2$ of G with $|V_1| = |V_2| = N/2$, we can find a partition with at least $\beta(N/2)^2$ edges between V_1 and V_2 . By Lemma 2.1 with $a = 1$, $t = 4n$, $d = n$, and $x = \epsilon^{-n}m$, there is a subset $A \subset V_2$ with cardinality at least $\frac{1}{2}\beta^{4n}|V_2| = \frac{1}{4}\beta^{4n}N$ such that for all but at most

$$2\beta^{-4n} \left(\frac{x}{N/2} \right)^{4n} \left(\frac{|A|}{N/2} \right) \binom{N/2}{n} \leq \left(\frac{2x}{\beta N} \right)^{4n} \frac{N^n}{n!} \quad (1)$$

subsets S_1 of A of size n , we have $|N_G(S_1)| \geq x$.

If G contains (not necessarily disjoint) subsets B_1, B_2 each of cardinality at least $2z$ such that every vertex in B_1 is adjacent to at least $(1 - \epsilon)|B_2|$ vertices in B_2 , then letting W_1 be any z vertices of B_1 and $W_2 = B_2 \setminus W_1$, we have a contradiction with the hypothesis of the lemma. Indeed, $|W_2| \geq |B_2|/2 \geq z$ and every vertex of W_1 is adjacent to at least $|W_2| - \epsilon|B_2| \geq (1 - 2\epsilon)|W_2|$ vertices in W_2 .

Let S_1 be a subset of A of cardinality n with $|N_G(S_1)| \geq x$. We will show that almost all subsets S_2 of A of cardinality n satisfy $|N_G(S_1) \cap N_{\bar{G}}(S_2)| \geq m$. The number of vertices u_1 of A such that

$$|N_G(S_1) \cap N_{\bar{G}}(u_1)| < \epsilon|N_G(S_1)|$$

is at most $2z$, otherwise each of these at least $2z$ vertices has at least $(1 - \epsilon)|N_G(S_1)|$ neighbors in $N_G(S_1)$, which by the above discussion would contradict the hypothesis of the lemma. Pick any vertex $u_1 \in A$ such that

$$|N_G(S_1) \cap N_{\bar{G}}(u_1)| \geq \epsilon|N_G(S_1)|.$$

After picking u_1, \dots, u_i such that

$$|N_G(S_1) \cap N_{\bar{G}}(\{u_1, \dots, u_i\})| \geq \epsilon^i |N_G(S_1)|,$$

again there are at most $2z$ vertices u_{i+1} such that

$$|N_G(S_1) \cap N_{\bar{G}}(\{u_1, \dots, u_i, u_{i+1}\})| < \epsilon^{i+1} |N_G(S_1)|,$$

otherwise each of these at least $2z$ vertices has at least $(1 - \epsilon)|N_G(S_1) \cap N_{\bar{G}}(\{u_1, \dots, u_i\})|$ neighbors in $N_G(S_1) \cap N_{\bar{G}}(\{u_1, \dots, u_i\})$, which by the above discussion would contradict the hypothesis of the lemma. Note that during this process for every index i there are at least $|A| - |S_1| - (i - 1) - 2z > |A| - 2z - 2n$ choices for $u_i \in A \setminus S_1$ not already chosen. Therefore, given S_1 with $|N_G(S_1)| \geq x$, we conclude that the number of ordered n -tuples (u_1, \dots, u_n) of distinct vertices of $A \setminus S_1$ with

$$|N_G(S_1) \cap N_{\bar{G}}(\{u_1, \dots, u_n\})| \geq \epsilon^n x = m$$

is at least

$$(|A| - 2z - 2n)^n \geq (|A| - 4z)^n \geq |A|^n - 4nz|A|^{n-1}.$$

Hence, the number of (unordered) subsets $S_2 = \{u_1, \dots, u_n\}$ of $A \setminus S_1$ with $|N_G(S_1) \cap N_{\bar{G}}(S_2)| < m$ is at most $4zn|A|^{n-1}/n!$. This implies that the number of disjoint pairs $S_1, S_2 \subset A$ with $|S_1| = |S_2| = n$, $|N_G(S_1)| \geq x$, and $|N_G(S_1) \cap N_{\bar{G}}(S_2)| < m$ is at most $\binom{|A|}{n} \cdot \frac{1}{n!} 4zn|A|^{n-1}$. Also, notice that by (1) the number of disjoint pairs $S_1, S_2 \subset A$ with $|S_1| = |S_2| = n$ and $|N_G(S_1)| < x$ is at most $\left(\frac{2x}{\beta N}\right)^{4n} \frac{N^n}{n!} \cdot \binom{|A|}{n}$.

Therefore, the number of pairs of disjoint subsets $S_1, S_2 \subset A$ with $|N_G(S_1) \cap N_{\bar{G}}(S_2)| < m$ and $|S_1| = |S_2| = n$ is at most

$$\left(\frac{2x}{\beta N}\right)^{4n} \frac{N^n}{n!} \binom{|A|}{n} + \binom{|A|}{n} \frac{1}{n!} 4zn|A|^{n-1}. \quad (2)$$

Using the upper bound on m and $|A| \leq N/2$, we have

$$\begin{aligned} \left(\frac{2x}{\beta N}\right)^{4n} \frac{N^n}{n!} \binom{|A|}{n} &\leq 2^{3n} \left(\frac{\epsilon^{-n}m}{\beta N}\right)^{4n} \frac{N^{2n}}{n!^2} = 2^{3n} \beta^{-4n} \epsilon^{-4n^2} \left(\frac{m}{N}\right)^{2n} \frac{m^{2n}}{n!^2} \\ &\leq 2^{3n} \beta^{-4n} \epsilon^{-4n^2} \left(\frac{\beta^2 \epsilon^{2n}}{16n}\right)^{2n} \frac{m^{2n}}{n!^2} = 2^{-5n} n^{-2n} \frac{m^{2n}}{n!^2} \\ &< \frac{1}{2} (2n)^{-2n} \binom{m}{n}^2. \end{aligned} \quad (3)$$

Using the lower bound on m and $|A| \leq N/2$, we have

$$\begin{aligned} \binom{|A|}{n} \frac{1}{n!} 4zn|A|^{n-1} &\leq n!^{-2} 4zn|A|^{2n-1} \leq 2^{3-2n} n!^{-2} n z N^{2n-1} \leq 2^{3-2n} n!^{-2} n \left(\frac{m}{4n}\right)^{2n} \\ &\leq n 2^{3-4n} (2n)^{-2n} \frac{m^{2n}}{n!^2} \leq \frac{1}{2} (2n)^{-2n} \binom{m}{n}^2. \end{aligned} \quad (4)$$

Combining (2), (3), and (4), we have that there are less than $(2n)^{-2n} \binom{m}{n}^2$ pairs of disjoint subsets $S_1, S_2 \subset A$ with $|S_1| = |S_2| = n$ and $|N_G(S_1) \cap N_{\bar{G}}(S_2)| < m$, completing the proof. \square

The next lemma follows from repeated application of Lemma 7.1 and an application of Corollary 6.2.

Lemma 7.2 *Let $\epsilon > 0$, H be a graph on $n \geq 3$ vertices and $G = (V, E)$ be an H -free graph with N vertices and no independent set of size t with $N \geq (4t)^{8n^3} \epsilon^{-4n^2} n$. Then there is a pair W_1, W_2 of disjoint subsets of V such that $|W_1|, |W_2| \geq (4t)^{-8n^3} \epsilon^{4n^2} N$ and every vertex in W_1 is adjacent to all but at most $2\epsilon|W_2|$ vertices of W_2 .*

Proof. Since G has no independent set of size t , then by Turán's theorem (see, e.g., [5],[14]) every induced subgraph of G with $v \geq t^2$ vertices has at least $\frac{v^2}{2t}$ edges. Let $z = (4t)^{-8n^3} \epsilon^{4n^2} N$, so $z \geq n$.

Suppose for contradiction that there are no disjoint subsets W_1, W_2 with $|W_1|, |W_2| \geq z$ and every vertex in W_1 adjacent to all but at most $2\epsilon|W_2|$ vertices of W_2 . Fix $\beta = \frac{1}{2t}$ and

$$m = \frac{\beta^2 \epsilon^{2n} (\frac{1}{4} \beta^{4n})^{n-1} N}{16n},$$

and repeatedly apply Lemma 7.1 $n - 1$ times (note that the choice of parameters allows this). We get a family of nested subsets $V = A_1 \supset \dots \supset A_n$ such that $|A_n| \geq (\frac{1}{4}\beta^{4n})^{n-1} N \geq m \geq 2n$ and for $1 \leq i \leq n - 1$, all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs (S_1, S_2) of disjoint subsets of A_{i+1} with $|S_1| = |S_2| = n$ have at least m vertices in A_i in the common neighborhood of S_1 in G and the common neighborhood of S_2 in \bar{G} . By Corollary 6.2, G contains H as an induced subgraph, contradicting the assumption that G is H -free, and completing the proof. \square

From Lemma 7.2, we quickly deduce Theorem 1.8, which says that for every H there is $c = c(H) > 0$ such that any H -free graph of order N contains a complete bipartite graph with parts of size N^c or an independent set of size N^c .

Proof of Theorem 1.8: Let H be a graph on n vertices, G be a H -free graph on N vertices, and $t = \frac{1}{10}N^{\frac{1}{10n^3}}$. If G has no independent set of size t , then by Lemma 7.2 with $\epsilon = \frac{1}{4t}$, G must contain disjoint subsets W_1 and W_2 each of cardinality at least $2t$ such that every vertex of W_1 is adjacent to all but at most $\frac{1}{2t}|W_2|$ vertices in W_2 . Picking t vertices in W_1 and their common neighborhood in W_2 , which has size at least $|W_2| - t\frac{1}{2t}|W_2| \geq |W_2|/2 \geq t$, shows that G contains $K_{t,t}$ and completes the proof. \square

We are now ready to prove Theorem 1.9, which says that for every H -free graph G of order N and n_1, n_2 satisfying $\log n_1 \log n_2 \leq c(H) \log N$, G contains a clique of size n_1 or an independent set of size n_2 . For a graph G , the clique number is the order of the largest complete subgraph of G and the independence number is the order of the largest independent set of G . Let $\omega_{t,n}(N)$ be the minimum clique number over all graphs with N vertices and independence number less than t that are not n -universal.

Proof of Theorem 1.9: Let $t = n_2$, H be a graph on n vertices, and G be an H -free graph with N vertices, no independent set of size n_2 , and clique number $\omega_{t,n}(N) < n_1$. Since \bar{G} is \bar{H} -free, we may suppose without loss of generality that $n_2 \geq n_1$. By Lemma 7.2 with $t = n_2$ and $\epsilon = \frac{1}{4t}$, there are disjoint subsets W_1 and W_2 of V , each of size at least $(4t)^{-8n^3-4n^2} N \geq (4t)^{-10n^3} N$, such that every vertex in W_1 is adjacent to all but at most $\frac{1}{2t}|W_2|$ vertices in W_2 . Pick a largest clique X in W_1 . The cardinality of clique X is less than $n_1 \leq n_2 = t$ by assumption. So $|X| < t$ and at least half of the vertices of W_2 are adjacent to X . Pick a largest clique Y in the vertices of W_2 adjacent to X . The clique number of G is at least $|X| + |Y|$. Hence,

$$\omega_{t,n}(N) \geq \omega_{t,n}(|W_1|) + \omega_{t,n}(|W_2|/2) \geq 2\omega_{t,n}((8t)^{-10n^3} N).$$

Let d be the largest integer such that $N \geq (8t)^{10n^3 d}$, so $d + 1 \geq \frac{1}{10n^3} \frac{\log N}{\log 8t}$. We have $\omega_{t,n}(N) \geq 2^d$ by repeated application of the inequality above. Hence,

$$\log n_1 \log n_2 \geq \log \omega_{t,n}(N) \log t \geq d \log t \geq \frac{1}{20n^3} \frac{\log N}{\log 8t} \log t \geq \frac{1}{80n^3} \log N,$$

completing the proof. \square

8 Edge distribution of H -free graphs

As we already mentioned in the introduction, there are several results which show that the edge distribution of H -free graphs is far from being uniform. One such result, obtained by Rödl, says that

for every graph H and $\epsilon \in (0, 1/2)$, there is a positive constant $\delta = \delta(\epsilon, H)$ such that any H -free graph on N vertices contains an induced subgraph on at least δN vertices with edge density either at most ϵ or at least $1 - \epsilon$. In [23], we gave an alternative proof which gives a much better bound on $\delta(\epsilon, H)$. Combining our techniques with the approach of [23], we obtain a generalization of Rödl's theorem which shows that a seemingly weak edge density condition is sufficient for an H -free graph to contain a very dense linear-sized induced subgraph. For $\delta \in (0, 1]$ and a monotone increasing function $\beta : (0, 1] \rightarrow (0, 1]$, we call a graph on N vertices (β, δ) -dense if every induced subgraph on σN vertices has edge density at least $\beta(\sigma)$ for $\sigma \geq \delta$.

Theorem 8.1 *For each monotone increasing function $\beta : (0, 1] \rightarrow (0, 1]$, $\epsilon > 0$, and graph H , there is $\delta = \delta(\beta, \epsilon, H) > 0$ such that every (β, δ) -dense H -free graph on n vertices contains an induced subgraph on at least δn vertices with edge density at least $1 - \epsilon$.*

Notice that Rödl's theorem is the special case of this statement when β is the constant function with value ϵ . An important step in the proof of Theorem 8.1 is the following lemma which shows how to find two large vertex subsets with edge density almost 1 between them in a (β, δ) -dense H -free graph.

Lemma 8.2 *Let $\beta : (0, 1] \rightarrow (0, 1]$ be a monotone increasing function, $\epsilon > 0$, and H be a graph on n vertices. There is $\delta > 0$ such that every H -free graph $G = (V, E)$ on N vertices that is (β, δ) -dense contains disjoint subsets $V_1, V_2 \subset V$ each of cardinality at least δN such that every vertex in V_1 is adjacent to all but at most $2\epsilon|V_2|$ vertices in V_2 .*

Proof. Define the sequence $\{\delta_i\}_{i=1}^n$ of real numbers in $(0, 1]$ recursively as follows: $\delta_1 = 1$ and $\delta_i = \frac{1}{4}\beta^{4n}(\delta_{i-1})\delta_{i-1}$. Let $\delta = (\frac{\epsilon^{2n}\delta_n}{64n^2})^{2n}n^{-1}$, $z = n\delta N$, and $m = \frac{\epsilon^{2n}\delta_n}{8n}N$, so

$$4nz^{\frac{1}{2n}}N^{1-\frac{1}{2n}} = m/2 \leq m = \frac{\epsilon^{2n}\delta_n}{8n}N = \frac{\epsilon^{2n}\beta^{4n}(\delta_{n-1})\delta_{n-1}}{32n}N \leq \frac{\beta^2(\delta_{n-1})\epsilon^{2n} \cdot \delta_{n-1}N}{16n}.$$

Since G is (β, δ) -dense, we have $N \geq \delta^{-1}$ so that $z \geq n$ and $m \geq 2n$.

Suppose for contradiction that G does not contain a pair V_1, V_2 of disjoint vertex subsets each of cardinality at least z such that every vertex in V_1 is adjacent to all but at most $2\epsilon|V_2|$ vertices in V_2 . By repeated application of Lemma 7.1 $n - 1$ times (note that the choice of parameters allows this), we find a family of nested subsets $V = A_1 \supset \dots \supset A_n$ with all $|A_i| \geq \delta_i N$ and $|A_n| \geq \delta_n N \geq m \geq 2n$ which have the following property. For all $i < n$, all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs (S_1, S_2) of subsets of A_{i+1} with $|S_1| = |S_2| = n$ have at least m vertices in A_i adjacent to all vertices in S_1 and no vertices in S_2 . By Corollary 6.2, G contains H as an induced subgraph, contradicting the assumption that G is H -free, and completing the proof. \square

The final step of the proof of Theorem 8.1 is to show how to go from two vertex subsets with edge density almost 1 between them as in Lemma 8.2 to one vertex subset with edge density almost 1. To accomplish this, we use the key lemma in [23]. We first need some definitions. For a graph $G = (V, E)$ and disjoint subsets $W_1, \dots, W_t \subset V$, the *density* $d_G(W_1, \dots, W_t)$ between the $t \geq 2$ vertex subsets W_1, \dots, W_t is defined by

$$d_G(W_1, \dots, W_t) = \frac{\sum_{i < j} e(W_i, W_j)}{\sum_{i < j} |W_i||W_j|},$$

where $e(A, B)$ is the number of pairs $(a, b) \in A \times B$ that are edges of G .

Definition 8.3 For $\alpha, \rho, \epsilon \in [0, 1]$ and positive integer t , a graph $G = (V, E)$ is $(\alpha, \rho, \epsilon, t)$ -dense if, for all subsets $U \subset V$ with $|U| \geq \alpha|V|$, there are disjoint subsets $W_1, \dots, W_t \subset U$ with $|W_1| = \dots = |W_t| = \lceil \rho|U| \rceil$ and $d_G(W_1, \dots, W_t) \geq 1 - \epsilon$.

By averaging, if $\alpha' \geq \alpha$, $\rho' \leq \rho$, $\epsilon' \geq \epsilon$, $t' \leq t$, and G is $(\alpha, \rho, \epsilon, t)$ -dense, then G is also $(\alpha', \rho', \epsilon', t')$ -dense. The key lemma in [23] (applied to the complement of the graph) says that if a graph is $(\frac{1}{2}\alpha\rho, \rho', \epsilon, t)$ -dense and $(\alpha, \rho, \epsilon/4, 2)$ -dense, then it is also $(\alpha, \frac{1}{2}\rho\rho', \epsilon, 2t)$ -dense.

Proof of Theorem 8.1: Fix a graph H on n vertices and a function $\beta : (0, 1] \rightarrow (0, 1]$. Note that if a graph G of order N is (β, δ) -dense, then, defining $\beta_\alpha(\sigma) = \beta(\alpha\sigma)$ for $0 < \alpha \leq 1$, every induced subgraph of G of size at least αN is $(\beta_\alpha, \alpha^{-1}\delta)$ -dense. Therefore, Lemma 8.2 implies that there is $\delta = \delta(\beta, \epsilon, H, \alpha)$ such that every (β, δ) -dense H -free graph is $(\alpha, \delta, \epsilon, 2)$ -dense.

We first show by induction on t that for $\alpha, \epsilon > 0$ and positive integer t , there is $\delta > 0$ such that every (β, δ) -dense H -free graph G is $(\alpha, \delta, \epsilon, 2^t)$ -dense. We have already established the base case $t = 1$. In particular, for $\alpha, \epsilon > 0$ there is $\delta' > 0$ such that every (β, δ') -dense H -free graph is $(\alpha, \delta', \epsilon/4, 2)$ -dense. Our induction hypothesis is that for $\alpha', \epsilon > 0$ there is $\delta^* > 0$ such that every (β, δ^*) -dense H -free graph G is $(\alpha', \delta^*, \epsilon, 2^{t-1})$ -dense. Letting $\alpha' = \frac{1}{2}\alpha\delta'$ and $\delta = \frac{1}{2}\delta'\delta^*$, then by the key lemma in [23] mentioned above, we have that every (β, δ) -dense H -free graph is $(\alpha, \delta, \epsilon, 2^t)$ -dense, which completes the induction.

If we use the last statement with $t = \log \frac{1}{\epsilon}$ and $\alpha = 1$, then we get that there are disjoint subsets $W_1, \dots, W_t \subset V$ with $t = \frac{1}{\epsilon}$, $|W_1| = \dots = |W_t| = \delta|V|$, and $d_G(W_1, \dots, W_t) \geq 1 - \epsilon$. Since $\binom{|W_1|}{2} \leq \frac{\epsilon}{t} \binom{t|W_1|}{2}$, then even if there are no edges in each W_i , the edge density in the set $W_1 \cup \dots \cup W_t$ is at least $1 - 2\epsilon$. Therefore, (using $\epsilon/2$ instead of ϵ) we have completed the proof of Theorem 8.1. \square

We use Theorem 8.1 in the next section to establish the results on disjoint edges in simple topological graphs. For the proof of Theorem 1.11, we need to know the dependence of δ on β in Theorem 8.1. Fix $\epsilon > 0$ and H , and let $\beta(\sigma) = \gamma\sigma$. A careful analysis of the proof of Lemma 8.2 demonstrates that there is a constant $c' = c'(\epsilon, H)$ such that in Lemma 8.2 we may take $\delta = \Omega(\gamma^{c'})$. Similarly, the above proof shows that there is a constant $c = c(\epsilon, H)$ such that in Theorem 8.1 we may take $\delta = \Omega(\gamma^c)$.

Rödl's theorem was extended by Nikiforov [39], who showed that if a graph has only few induced copies of H , then it can be partitioned into a constant number of sets each of which is either very sparse or very dense. We would like to remark that our proof can be easily modified to give a similar extension of Theorem 8.1, which shows that a (β, δ) -dense graph with few induced copies of H has a partition into a constant number of very dense subsets.

9 Edge intersection patterns in simple topological graphs

We next provide details of the proofs of Theorem 1.10 and Corollary 1.11 on disjoint edge patterns in simple topological graphs. We first need to establish an analogue of the well-known Crossing Lemma which states that every simple topological graph with n vertices and $m \geq 4n$ edges contains at least $\frac{m^3}{64n^2}$ pairs of crossing edges. Using the proof of this lemma (see, e.g., [5]) together with the linear upper bound on the number of edges in a thrackle, it is straightforward to obtain a similar result for disjoint edges in simple topological graphs. For the sake of completeness, we sketch the proof here.

Lemma 9.1 Every simple topological graph $G = (V, E)$ with n vertices and $m \geq 2n$ edges has at least $\frac{m^3}{16n^2}$ pairs of disjoint edges.

Sketch of Proof: Let t be the number of disjoint edges in G . The result in [9] that every n -vertex simple topological graph without a pair of disjoint edges has at most $\frac{3}{2}(n-1)$ edges implies that every n -vertex simple topological graph with m edges has at least $m - \frac{3}{2}(n-1) \geq m - \frac{3}{2}n$ pairs of disjoint edges. Let G' be the random induced subgraph of G obtained by picking each vertex with probability $p = 2n/m \leq 1$. The expected number of vertices of G' is pn , the expected number of edges of G' is p^2n , and the expected number of pairs of disjoint edges in the given embedding of G' is p^4t . Hence, $p^4t \geq p^2m - \frac{3}{2}pn$, or equivalently, $t \geq p^{-2}m - \frac{3}{2}p^{-3}n = \frac{m^3}{16n^2}$, which is the desired result. \square

Another ingredient in the proof of Theorem 1.10 is a separator theorem for curves proved in [22]. A *separator* for a graph $\Gamma = (V, E)$ is a subset $V_0 \subset V$ such that there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq \frac{2}{3}|V|$ and no vertex in V_1 is adjacent to any vertex in V_2 . Using the well-known Lipton-Tarjan separator theorem for planar graphs, Fox and Pach [22] proved that the intersection graph of any collection of curves in the plane with k crossings has a separator of size at most $C\sqrt{k}$, where C is an absolute constant. Recall that Theorem 1.10 says that for each $\gamma > 0$ there is $\delta > 0$ and n_0 such that every simple topological graph $G = (V, E)$ with $n \geq n_0$ vertices and $m \geq \gamma n^2$ edges contains two disjoint edge subsets E_1, E_2 each of cardinality at least δn^2 such that every edge in E_1 is disjoint from every edge in E_2 .

Proof of Theorem 1.10: Define an auxiliary graph Γ with a vertex for each edge of the simple topological graph G in which a pair of vertices of Γ are adjacent if and only if their corresponding edges in G are disjoint. Lemma 9.1 tells us that every induced subgraph of Γ with $\sigma m \geq \sigma \gamma n^2 \geq 2n$ vertices has at least $\frac{(\sigma m)^3}{16n^2}$ edges and therefore has edge density at least

$$\frac{(\sigma m)^3}{16n^2} / \binom{\sigma m}{2} \geq \frac{\sigma m}{8n^2} \geq \frac{\gamma \sigma}{8}.$$

In other words, Γ is (β, δ) -dense with $\beta(\sigma) = \frac{\gamma \sigma}{8}$ and $\delta = \frac{2}{\gamma n}$. Let H be the 15-vertex graph which is the complement of the 1-subdivision of K_5 . As mentioned in Section 1.5, the intersection graph of curves in the plane does not contain the 1-subdivision of K_5 as an induced subgraph and therefore the graph Γ is H -free. Hence, Theorem 8.1 implies that for each $\epsilon > 0$ there is $\delta' > 0$ and an induced subgraph Γ' of Γ with order at least $\delta' m \geq \delta' \gamma n^2$ and edge density at least $1 - \epsilon$. We use this fact with $\epsilon = \frac{1}{36C^2}$, where C is the constant in the separator theorem for curves. Since Γ' has edge density at least $1 - \epsilon$ and each pair of edges in the simple topological graph cross at most once, then the number k of crossings between edges of G corresponding to vertices of Γ' is less than $\epsilon |\Gamma'|^2 = \frac{1}{36C^2} |\Gamma'|^2$. Applying the separator theorem for curves, we get a partition of the vertex set of Γ' into subsets V_0, V_1, V_2 with $|V_0| \leq C\sqrt{\frac{1}{36C^2} |\Gamma'|^2} \leq |\Gamma'|/6$ and $|V_1|, |V_2| \leq 2|\Gamma'|/3$, and no edges in Γ' between V_1 and V_2 . In particular, both V_1 and V_2 have cardinality at least $|\Gamma'|/6$. Therefore, letting $\delta = \frac{1}{6}\delta'\gamma$, we have two edge subsets E_1, E_2 of G (which correspond to V_1, V_2 in Γ') each with cardinality at least δn^2 such that every edge in E_1 is disjoint from every edge in E_2 . \square

As we already mentioned in the discussion right after the proof of Theorem 8.1, the value of δ' which was used in the above proof of Theorem 1.10 satisfies $\delta' \geq \gamma^{c'}$ for some constant c' . Since $\delta = \frac{1}{6}\delta'\gamma \geq \frac{1}{6}\gamma^{c'+1}$, we have the following quantitative version of Theorem 1.10. There is a constant c such that every simple topological graph $G = (V, E)$ with n vertices and at least γn^2 edges with $\gamma \geq 2/n$ has two disjoint edge subsets $E_1, E_2 \subset E$ each of size at least $\gamma^c n^2$ such that every edge in E_1 is disjoint from every edge in E_2 .

We next prove a strengthening of Corollary 1.11. It says that any simple topological graph on n vertices and at least γn^2 edges contains $\gamma'(\log n)^{1+a}$ disjoint edges where $\gamma' > 0$ only depends on γ

and $a > 0$ is an absolute constant.

Proof of Corollary 1.11: Let d be the largest positive integer such that $\gamma^{c^d} \geq n^{-1/2}$, where c is the constant in the quantitative version of Theorem 1.10 stated above. By repeated application of this quantitative version, we get disjoint subsets E_1, \dots, E_{2^d} each of size at least $\gamma^{c^d} n^2 \geq n^{3/2}$ such that no edge in E_i intersects an edge in E_j for all $i \neq j$. By definition of d , we have $\gamma^{c^{d+1}} < n^{-1/2}$, which implies that $2^d \geq \left(\frac{\log n}{2c \log 1/\gamma}\right)^{1/\log c} = \gamma_1 (\log n)^b$ where $\gamma_1 > 0$ only depends on γ and $b = 1/\log c > 0$ is an absolute constant. Now we need to use the result of Pach and Tóth [41] mentioned in Section 1.5, which says that every simple topological graph of order n without k pairwise disjoint edges has $O(n(\log n)^{4k-8})$ edges. By choosing $k' = \frac{\log n}{8 \log \log n}$, we conclude that every simple topological graph with at least $n^{3/2}$ edges (in particular, each of the sets E_i) contains at least k' pairwise disjoint edges. Therefore, altogether G contains $\gamma_1 (\log n)^b \cdot \frac{\log n}{8 \log \log n} \geq \gamma' (\log n)^{1+a}$ pairwise disjoint edges, where $\gamma' > 0$ only depends on γ and $a > 0$ is any absolute constant less than b . \square

10 Monochromatic Induced Copies

The goal of this section is to prove the upper bound on multicolor induced Ramsey numbers in Theorem 1.12. To accomplish this, we demonstrate that the graph Γ which gives the bound in this theorem can be taken to be any pseudo-random graph of appropriate order and edge density. Recall that the *random graph* $G(n, p)$ is the probability space of labeled graphs on n vertices, where every edge appears independently with probability p . An important property of $G(n, p)$ is that, with high probability, between any two large subsets of vertices A and B , the edge density $d(A, B)$ is approximately p , where $d(A, B)$ is the fraction of ordered pairs $(a, b) \in A \times B$ that are edges. This observation is one of the motivations for the following useful definition. A graph $\Gamma = (V, E)$ is (p, λ) -pseudo-random if the following inequality holds for all (not necessarily disjoint) subsets $A, B \subset V$:

$$|d(A, B) - p| \leq \frac{\lambda}{\sqrt{|A||B|}}.$$

The survey by Krivelevich and Sudakov [37] contains many examples of (p, λ) -pseudo-random graphs on n vertices with $\lambda = O(\sqrt{pn})$. One example is the random graph $G(n, p)$ which with high probability is (p, λ) -pseudo-random with $\lambda = O(\sqrt{pn})$ for $p < .99$. The Paley graph P_N is another example of a pseudo-random graph. For N a prime power, P_N has vertex set \mathbb{F}_N and distinct elements $x, y \in \mathbb{F}_N$ are adjacent if $x - y$ is a square. It is well known (see, e.g., [37]) that the Paley graph P_N is $(1/2, \sqrt{N})$ -pseudo-random. We deduce Theorem 1.12 from the following theorem.

Theorem 10.1 *If $n, k \geq 2$ and Γ is (p, λ) -pseudo-random with N vertices, $0 < p \leq 1/2$, and $\lambda \leq (k/p)^{-100n^3k} N$, then every graph on n vertices has a monochromatic induced copy in every k -edge-coloring of Γ . Moreover, all of these monochromatic induced copies can be found in the same color.*

By letting Γ be a sufficiently large, pseudo-random graph with $p = 1/2$, Theorem 1.12 follows from Theorem 10.1. For example, with high probability, the graph Γ can be taken to be the random graph $G(N, 1/2)$ with $N = k^{500n^3k}$. Alternatively, for an explicit construction, we can take Γ to be a Paley graph P_N with $N \geq k^{500n^3k}$ prime.

The following lemma is the main tool in the proof of Theorem 10.1. In the setting of Lemma 10.2, we have a graph G that is a subgraph of a pseudo-random graph Γ . We use Lemma 2.1 to show there

is a large subset A of vertices such that $|N_G(S_1)|$ is large for almost all small subsets S_1 of A . We use the pseudo-randomness of Γ to ensure that for almost all small disjoint subsets S_1 and S_2 of A , there are many vertices adjacent to S_1 in G and adjacent to S_2 in $\bar{\Gamma}$.

Lemma 10.2 *Let Γ be a (p, λ) -pseudo-random graph with $p \leq 1/2$ and G be a subgraph with order N and $\epsilon \binom{N}{2}$ edges. Suppose m and n are positive integers such that*

$$8n(\lambda/p)^{\frac{2}{2n+1}} N^{1-\frac{2}{2n+1}} < m < \frac{\epsilon^2}{2^{10pn+4}n} N.$$

Then there is a subset $A \subset V$ with $|A| \geq \frac{1}{4}\epsilon^{4n}N$ such that for all but less than $(2n)^{-2n}\binom{m}{n}^2$ pairs of disjoint subsets $S_1, S_2 \subset A$ with $|S_1| = |S_2| = n$, there are at least m vertices adjacent to every vertex of S_1 in G and no vertex of S_2 in Γ .

Proof. By averaging over all partitions $V = V_1 \cup V_2$ of G with $|V_1| = |V_2| = N/2$, we can find a partition with at least $\epsilon(N/2)^2$ edges between V_1 and V_2 . By Lemma 2.1 with $a = 1$, $t = 4n$, $d = n$, and $x = (1 - 3p/2)^{-n}m$, there is a subset $A \subset V_1$ with cardinality at least $\frac{1}{2}\epsilon^{4n}|V_1| = \frac{1}{4}\epsilon^{4n}N$ such that the number of subsets S_1 of A of size n with $|N_G(S_1)| < x$ is at most

$$2\epsilon^{-4n} \left(\frac{x}{N/2}\right)^{4n} \left(\frac{|A|}{N/2}\right) \binom{N/2}{n} \leq \left(\frac{2x}{\epsilon N}\right)^{4n} \binom{N}{n} \leq \epsilon^{-4n} 2^{20pn^2+4n} (m/N)^{4n} \binom{N}{n},$$

where the last inequality follows from the simple inequality $1 - 3p/2 \geq 2^{-5p}$ for $p \leq 1/2$. This implies, using the upper bound on m , that the number of disjoint pairs S_1, S_2 of subsets of A with $|S_1| = |S_2| = n$ and $|N_G(S_1)| < x$ is at most

$$\begin{aligned} \epsilon^{-4n} 2^{20pn^2+4n} (m/N)^{4n} \binom{N}{n} \cdot \binom{N}{n} &\leq \epsilon^{-4n} 2^{20pn^2+4n} (m/N)^{2n} \frac{m^{2n}}{n!^2} \\ &\leq \epsilon^{-4n} 2^{20pn^2+4n} \left(\frac{\epsilon^2}{2^{10pn+4}n}\right)^{2n} \frac{m^{2n}}{n!^2} \leq 2^{-4n} n^{-2n} \frac{m^{2n}}{n!^2} \\ &< \frac{1}{2} (2n)^{-2n} \binom{m}{n}^2. \end{aligned} \tag{5}$$

Let S_1 be a subset of A of cardinality n with $|N_G(S_1)| \geq x = (1 - 3p/2)^{-n}m$. We will show that almost all subsets S_2 of $A \setminus S_1$ of cardinality n satisfy $|N_G(S_1) \cap N_{\bar{\Gamma}}(S_2)| \geq m$. To do this, we give a lower bound on the number of ordered n -tuples (u_1, \dots, u_n) of distinct vertices of $A \setminus S_1$ such that for each i , $|N_G(S_1) \cap N_{\bar{\Gamma}}(\{u_1, \dots, u_i\})| \geq (1 - 3p/2)^i |N_G(S_1)|$. Suppose we have already picked u_1, \dots, u_{i-1} satisfying $|N_G(S_1) \cap N_{\bar{\Gamma}}(\{u_1, \dots, u_{i-1}\})| \geq (1 - 3p/2)^{i-1} |N_G(S_1)| \geq m$. Let X_i denote the set of vertices u_i in A with $|N_G(S_1) \cap N_{\bar{\Gamma}}(\{u_1, \dots, u_i\})| < (1 - 3p/2)^i |N_G(S_1)|$. Then the edge density between X_i and $N_G(S_1) \cap N_{\bar{\Gamma}}(\{u_1, \dots, u_{i-1}\})$ in Γ is more than $3p/2$. Since Γ is (p, λ) -pseudo-random, we have

$$p/2 < \frac{\lambda}{\sqrt{|X_i| \cdot |N_G(S_1) \cap N_{\bar{\Gamma}}(\{u_1, \dots, u_{i-1}\})|}} \leq \frac{\lambda}{\sqrt{|X_i|m}}.$$

Therefore, $|X_i| < 4(\lambda/p)^2 m^{-1}$. Hence, during this process for every index i there are at least

$$|A| - |S_1| - (i-1) - \frac{4(\lambda/p)^2}{m} > |A| - 2n - 4(\lambda/p)^2 m^{-1} \geq |A| - 8n(\lambda/p)^2 m^{-1}$$

choices for $u_i \in A \setminus (S_1 \cup \{u_1, \dots, u_{i-1}\} \cup X_i)$. Therefore, given S_1 with $|N_G(S_1)| \geq x$, we conclude that the number of ordered n -tuples (u_1, \dots, u_n) of distinct vertices of $A \setminus S_1$ with

$$|N_G(S_1) \cap N_{\bar{\Gamma}}(\{u_1, \dots, u_n\})| \geq (1 - 3p/2)^n |N_G(S_1)| \geq m$$

is at least

$$(|A| - 8n(\lambda/p)^2 m^{-1})^n \geq |A|^n - 8n^2(\lambda/p)^2 m^{-1} |A|^{n-1}.$$

This together with the lower bound on m implies that the number of pairs S_1, S_2 of disjoint (unordered) subsets of A with $|N_G(S_1)| \geq x$ and $|N_G(S_1) \cap N_{\bar{\Gamma}}(S_2)| < m$ is at most

$$\begin{aligned} \binom{N}{n} \cdot \frac{1}{n!} 8n^2(\lambda/p)^2 m^{-1} |A|^{n-1} &\leq 8n^2 m^{-1} n!^{-2} (\lambda/p)^2 N^{2n-1} \leq 8n^2 m^{-1} n!^{-2} \left(\frac{m}{8n}\right)^{2n+1} \\ &= 2^{-4n} n (2n)^{-2n} \frac{m^{2n}}{n!^2} < \frac{1}{2} (2n)^{-2n} \binom{m}{n}^2. \end{aligned} \quad (6)$$

Combining (5) and (6), all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs S_1, S_2 of disjoint subsets of A with $|S_1| = |S_2| = n$ satisfy $|N_G(S_1) \cap N_{\bar{\Gamma}}(S_2)| \geq m$, which completes the proof. \square

We are now ready to prove our main result in this section.

Proof of Theorem 10.1: Consider a k -edge-coloring of the (p, λ) -pseudo-random graph Γ with colors $1, \dots, k$. Let B_1 denote the set of vertices of Γ . For $j \in \{1, \dots, k\}$, let G_j denote the graph of color j . Let $\epsilon = \frac{p}{2k}$ and $m = \epsilon^{20n^2k} N$.

We will pick nested subsets $B_1 \supset \dots \supset B_{k(n-2)+2}$ such that, for each $i \leq k(n-2)+1$, we have $|B_{i+1}| \geq \frac{1}{4} \epsilon^{4n} |B_i|$ and there is a color $c(i) \in \{1, \dots, k\}$ such that all but less than $(2n)^{-2n} \binom{m}{n}^2$ pairs of disjoint subsets $S_1, S_2 \subset B_{i+1}$ each of size n have at least m vertices in B_i adjacent to S_1 in $G_{c(i)}$ and adjacent to S_2 in $\bar{\Gamma}$. Once we have found such a family of nested subsets, the proof is easy. By the pigeonhole principle, one of the k colors is represented at least $n-1$ times in the sequence $c(1), \dots, c(k(n-2)+1)$. We suppose without loss of generality that 1 is this popular color. Let $i(1) = 1$ and for $1 < j \leq n-1$, let $i(j)$ be the j^{th} smallest integer $i > 1$ such that $c(i-1) = 1$. Letting $A_j = B_{i(j)}$, we have, by Lemma 6.1 with G_1 as G and $\bar{\Gamma}$ as F , that there is an induced copy of every graph on n vertices that is monochromatic of color 1. So, for the rest of the proof, we only need to show that there are nested subsets $B_1 \supset \dots \supset B_{k(n-2)+2}$ and colors $c(1), \dots, c(k(n-2)+1)$ with the desired properties.

We now show how to pick $c(i)$ and B_{i+1} having already picked B_i . Let $c(i)$ denote the densest of the k colors in $\Gamma[B_i]$. By pseudo-randomness of Γ , it is straightforward to check that the density of Γ in B_i is at least $p/2$, so the edge density of color $c(i)$ in $G[B_i]$ is at least $\frac{p}{2k} = \epsilon$. Indeed, if not, then the density between B_1 and itself in Γ deviates from p by at least $p/2$ and so, by pseudo-randomness of Γ ,

$$\frac{2\lambda}{p} \geq |B_i| \geq \left(\frac{1}{4} \epsilon^{4n}\right)^i N \geq \left(\frac{p}{4k}\right)^{4n^2k} N \geq (p/k)^{20n^2k} N,$$

contradicting the upper bound on λ . Since $m = \epsilon^{20n^2k} N$, $k \geq 2$, $p \leq 1/2$, and $\epsilon = \frac{p}{2k} \leq 1/8$, we have

$$\begin{aligned} 8n(\lambda/p)^{\frac{2}{2n+1}} |B_i|^{1-\frac{2}{2n+1}} &\leq 8n \left((k/p)^{-100n^3k} N/p \right)^{\frac{2}{2n+1}} N^{1-\frac{2}{2n+1}} < 8n(k/p)^{-50n^2k} N \\ &< m < \frac{\epsilon^2}{2^{20n}} \left(\frac{1}{4} \epsilon^{4n} \right)^{nk} N < \frac{\epsilon^2}{2^{10pn+4n}} |B_i|. \end{aligned}$$

Hence, we may apply Lemma 10.2 to the graph $G_{c(i)}[B_i]$, which is a subgraph of the (p, λ) -pseudo-random graph Γ , and get a subset B_{i+1} of B_i with the desired properties. This completes the proof of the induction step and the proof of theorem. \square

11 Concluding Remarks

- We conjecture that there is an absolute constant c such that $r(H) \leq 2^{c\Delta}n$ for every H with n vertices and maximum degree Δ and our results confirm it for graphs of bounded chromatic number. This question is closely related to another old problem on Ramsey numbers. More than thirty years ago, Erdős conjectured that $r(H) \leq 2^{c\sqrt{m}}$ for every graph H with m edges and no isolated vertices. The best known bound for this question is $r(H) \leq 2^{c\sqrt{m}\log m}$ (see [3]) and the solution of our conjecture might lead to further progress on the problem of Erdős as well.
- The bound $r(H) \leq 2^{4d+12}\Delta n$ for bipartite d -degenerate n -vertex graphs with maximum degree Δ shows that $\log r(H) \leq 4d + 2\log n + 12$. On the other hand, the standard probabilistic argument gives the lower bound $r(H) \geq \max(2^{d(H)/2}, n)$, where the *degeneracy number* $d(H)$ is the smallest d such that H is d -degenerate. It therefore follows that $\log r(H) = \Theta(d(H) + \log n)$ for every bipartite graph H (this can be also deduced with slightly weaker constants from the result in [3]). In particular, since $d(H)$ can be quickly computed by simply deleting the vertex with minimum degree and repeating this until the graph is empty, we can efficiently compute $\log r(H)$ up to a constant factor for every bipartite graph H . It is plausible that $\log r(H) = \Theta(d(H) + \log n)$ for every d -degenerate n -vertex graph H . If so, then this would imply the above mentioned conjecture of Erdős that $r(H) \leq 2^{c\sqrt{m}}$ for every graph H with m edges and no isolated vertices since every such graph satisfies $d(H) = O(\sqrt{m})$.
- The exciting conjecture of Burr and Erdős, which was the driving force behind most of the research done on Ramsey numbers for sparse graphs, is still open. Moreover, we even do not know how to deal with the interesting special case of bipartite graphs in which all vertices in one part have bounded degree. However, the techniques in this paper can be used to make modest progress and solve this special case when the bipartite graph H is bi-regular, i.e., every vertex in one part has degree Δ_1 and every vertex in the other part has degree Δ_2 . The proof of the following theorem is a minor variation of the proof of Theorem 1.1 and therefore is omitted.

Theorem 11.1 *Let $H = (V_1, V_2)$ be a bipartite graph without isolated vertices such that, for $i \in \{1, 2\}$, the number of vertices in V_i is n_i and the maximum degree of a vertex in V_i is Δ_i . Then $r(H) \leq 2^{c\Delta_1}\Delta_2n_2$ for some absolute constant c .*

Note that this theorem implies that if H also satisfies $\Delta_2n_2 = 2^{O(\Delta_1)}n$, then $r(H) = 2^{O(\Delta_1)}n$, where n is the number of vertices of H . In particular, this bound is valid for graphs whose average degree in each part is at least a constant fraction of the maximum degree in that part.

Also, it is possible to extend ideas used in the proofs of Theorems 1.4 and 1.5 to show that for every $0 < \delta \leq 1$ the Ramsey number of any d -degenerate graph H with n vertices and maximum degree Δ satisfies $r(H) \leq 2^{c/\delta}\Delta^\delta n$, where c is a constant depending only on d . By taking $\delta = (\log n)^{-1/2}$ we have that $r(H) \leq 2^{c(d)\sqrt{\log n}}n$ for every d -degenerate graph of order n . This improves the result in [35].

- One should be able to extend the bound in Theorem 1.10 to work for all possible sizes of simple topological graphs. Moreover, it might be true that every simple topological graph with $m = \epsilon n^2$ edges with $\epsilon \geq 2/n$ contains two sets of size δn^2 of pairwise disjoint edges with $\delta = c\epsilon^2$ for some absolute constant $c > 0$. This would give both Theorem 1.10 and, taking $\epsilon = 2/n$, a linear bound on the size of thrackles. For comparison, our proof of Theorem 1.10 demonstrates that δ can be taken to be a polynomial in ϵ .

It would be also interesting to extend Conway’s conjecture by showing that for every fixed k , the number of edges in a simple topological graph with n vertices and no k pairwise disjoint edges is still linear in n . This is open even for $k = 3$, though (see Section 1.5) an almost linear upper bound was given in [41]. For geometric graphs, such a linear bound was a longstanding conjecture of Erdős and Perles and was settled in the affirmative by Pach and Tőrocsik.

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Note added in proof. After this paper was written we learned that D. Conlon proved the following variant of Corollary 1.2, independently and simultaneously with our work. He showed that $r(H) \leq 2^{(2+o(1))\Delta} n$ for bipartite n -vertex graph H with maximum degree Δ .

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